UNIT - IV

12/9/19

UNIT - 3

Stochastic Process Or Random Process

Temporal Characteristics

> Stochastic or Random Process:

(i) Explain random process with neat sketch.

> Classifications of random process:

There are 4 classification (i) continuous random process
(ii) discrete random process
(iii) Continuous random sequence
(iv) discrete random sequence.

(ii) Explain classifications of random process with neat sketch.

> Deterministic & Non-deterministic Random Process:

(iii) Explain deterministic & non-deterministic random process with example.

(In materical).

> Statistical Properties of Random Process:

(iv) Explain types of statistical averages.

(a) Mean: The mean value of random process $X(t)$, is the expected value of the random process $X(t)$, it is denoted as $\overline{X(t)}$

The mean value of $X(t) = \overline{X(t)} = E[X(t)] = \int_{-\infty}^{\infty} x f_X(x;t)\, dx$

where $f_X(x;t)$ is the PDF of random process $X(t)$

The mean of random process is also called as Ensemble Average of random process.

(b) Auto-Correlation: Consider a random process $X(t)$. let X_1 and X_2 are two random variables defined at time instant t_1 and t_2, respectively, with joint density function $f_X(x_1, x_2 : t_1, t_2)$.

Contd

The auto-correlation function of X_1 & X_2 = $R_{XX}(t_1, t_2)$

$$= E[X_1 X_2] = E[X(t_1) \cdot X(t_2)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2 : t_1, t_2)\, dx_1\, dx_2$$

(c) **Cross-Correlation:** Consider two random processes $X(t)$ and $Y(t)$ defined with random variables X and Y at time instants t_1 and t_2, respectively, with the joint density function $f_X(x, y : t_1, t_2)$

The cross-correlation function of X & Y is

$$R_{XY}(t_1, t_2) = E[XY] = E[X(t_1) Y(t_2)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x, y : t_1, t_2)\, dx\, dy$$

> **Stationary Random Process:**

A random process is said to be stationary if all its statistical properties such as mean, moments, variances, correlations etc, do not change with time.

(a) **First order stationary random process:**

A random process is said to be first order stationary if its first order density function doesnot change with time or shift in the time value.

If $X(t)$ is the first order stationary random process then

$$f_X(x_1 : t_1) = f_X(x_1 : t_1 + \Delta t) \quad \forall\, t\ \&\ \Delta t$$

where Δt is shift in the time value. Since $f_X(x_1 : t_1)$ is independent of time(t_1), the mean value of the process is constant.

Therefore the condition for first order stationary random process is its mean value is constant.

$$\boxed{E[X(t)] = \bar{X} = \text{constant}}$$

Proof: Consider a random process $X(t)$ with random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ at time instants t_1 and $t_2 = t_1 + \Delta t$

$$E[X_1] = E[X(t_1)] = \int_{-\infty}^{\infty} x_1 f_X(x_1 : t_1) \cdot dx_1$$

$$E[X_2] = E[X(t_2)] = \int_{-\infty}^{\infty} x_2 f_X(x_2 : t_2)\, dx_2$$

Put $x_2 = x_1 \Rightarrow dx_2 = dx_1$

$$= \int_{-\infty}^{\infty} x_1 f_X(x_1 : t_2)\, dx_1$$

let $t_2 = t_1 + \Delta t$

$$= \int_{-\infty}^{\infty} x_1 f_X(x_1 : t_1 + \Delta t)\, dx_1$$

$\because f_X(x_1 : t_1) = f_X(x_1 : t_1 + \Delta t)$ for stationary random process.

$$E[X_2] = E[X(t_2)] = \int_{-\infty}^{\infty} x_1 f_X(x_1 : t_1)\, dx_1$$

$$E[X(t_2)] = E[X(t_1)]$$

$$E[X(t_1 + \Delta t)] = E[X(t)] = \text{constant}$$

(b) Second order stationary random process:

A random process is said to be second order stationary random process if its second order joint density function do not change with time i.e.,

$$f_X(x_1, x_2 : t_1, t_2) = f_X(x_1, x_2 : t_1 + \Delta t, t_2 + \Delta t)$$

$\forall\ t_1, t_2\ \&\ \Delta t$.

It is a function of time difference $(t_2 - t_1)$ not absolute time t.

Condition: The condition for a second order stationary process is its auto-correlation function depends only on time differences and not on absolute time i.e., if X_1 and X_2 are the two r.v's of random process $X(t)$ defined at t_1 & t_2 then the auto-correlation function is $R_{XX}(t_1, t_2) = E[X(t_1)\, X(t_2)]$

If $\tau = t_2 - t_1 \Rightarrow t_2 = \tau + t_1$

$$R_{XX}(t_1, t_1 + \tau) = E[X_1(t)\; X(\tau + t_1)]$$

let $t_1 = t$

$$\boxed{R_{XX}(t_1, t + \tau) = E[(X(t)\; X(t+\tau)] = R_{XX}(\tau)}$$

Note: All second order stationary process is also first order stationary process.

(C) Wide Sense Stationary Random Process:

If a random process $X(t)$ is a second order stationary process, then it is called wide sense stationary (WSS) process or Weak Sense stationary random process. However, the converse is not true.

The conditions for widesense stationary process are

(i) $E[X(t)] = \bar{X} = \text{constant}$

(ii) $R_{XX}(t, t+\tau) = E[X(t)\; X[t+\tau]] = R_{XX}(\tau) \rightarrow$ is independent of time 't'.

(d) Jointly Wide Sense Stationary Random Process:

Consider two random processes $X(t)$ and $Y(t)$, if they are jointly wide sense stationary, then the cross-correlation function of $X(t)$ and $Y(t)$ is function of time difference $\tau = t_2 - t_1$ only and not absolute time t i.e., the cross-correlation function

$$R_{XY}(t_1, t_2) = E[X(t_1)\, Y(t_2)]$$ if the time differen

$\tau = t_2 - t_1 \Rightarrow t_1 = t$ then

$$R_{XY}(t_1, t_1+\tau) = E[X(t_1)\, Y(t_1+\tau)]$$

$$R_{XY}(t, t+\tau) = E[X(t)\, Y(t+\tau)] = R_{XY}(\tau)$$

$\therefore$ Jointly wide sense stationary's conditions are

(i) $E[X(t)] = \bar{X} = \text{constant}$

(ii) $E[Y(t)] = \bar{Y} = \text{constant}$

(iii) $R_{XY}(t, t+\tau) = E[X(t)\, Y(t+\tau)] = R_{XY}(\tau)$ independent time 't'.

(e) Strict Sense Stationary (SSS) Random process:

A random process $X(t)$ is said to strict sense stationary if its N^{th} order joint density function doesnot change with time i.e., $f_X(x_1, x_2 \ldots x_N : t_1, t_2 \ldots t_N)$

$$= f_X(x_1, x_2 \ldots x_N : t_1+\Delta t, t_2+\Delta t, \ldots t_N+\Delta t)$$

$$\forall\ t_1, t_2, \ldots t_N \ \& \ \Delta t.$$

It is also seen that a process is stationary to all orders $n = 1, 2 \ldots N$ is called strict sense stationary random process. The SSS process is also called N^{th} order stationary process.

Note:

SSS process is also wide sense stationary process but converse is not true.

(V) Explain stationary random process.

(Vi) Differentiate b/w WSS and SSS process.

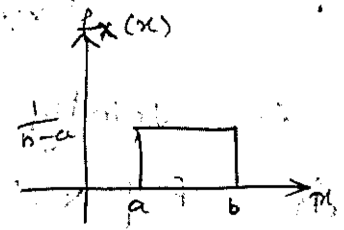
* Problems on SSS process:

1. A random process is given as $X(t) = A.t$ where A is an uniformly distributed random variable on $(0, 2)$. Find whether $X(t)$ is WSS or not.

Sol: Given random process $X(t) = A.t$

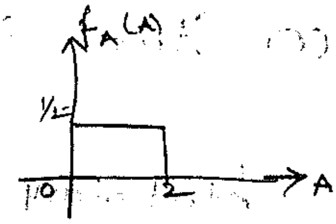
We know density function of uniformly distributed random variable X on (a, b) i.e.,

$$f_X(x) = \begin{cases} \frac{1}{b-a} & ; a \leq x \leq b \\ 0 & ; \text{elsewhere} \end{cases}$$



For the given problem, A is uniformly distributed on $(0, 2)$ which is a random variable, then the density function is defined by

$$f_A(A) = \begin{cases} \frac{1}{2} & ; 0 \leq A \leq 2 \\ 0 & ; \text{elsewhere} \end{cases}$$



We know condition for WSS process,

(i) $E[X(t)] = \overline{X} = \text{constant}$

(ii) $R_{XX}(t, t+\tau) = E[X(t)\,X(t+\tau)] = R_{XX}(\tau)$ - is not a function of time 't'.

(i) The mean value of $X(t) = E[X(t)]$

$$\overline{X} = \int_{-\infty}^{\infty} X(t)\, f_A(A)\, dA$$

$$= \int_{-\infty}^{0} X(t)(0)\, dA + \int_{0}^{2} A t \cdot \frac{1}{2}\, dA + \int_{2}^{\infty} X(t)(0)\, dA$$

$$= \int_0^2 \frac{1}{2} At \, dA$$

$$= \frac{t}{2} \left[\frac{A^2}{2}\right]_0^2$$

$$= \frac{t}{\cancel{2}} \left[\frac{\cancel{4}^{2}}{\cancel{2}} - 0\right]$$

$$= t$$

$$\neq \text{constant}$$

$$\therefore \quad E[x(t)] = t \neq \text{constant}$$

(ii) Autocorrelation function of $X(t)$:

$$R_{XX}(t, t+\tau) = E[x(t)\, x(t+\tau)]$$

$$= R_{XX}(\tau)$$

$$= \int_{-\infty}^{\infty} f_A(A) \cancel{dA} \cdot x(t)\, x(t+\tau) \, dA$$

$$= \int_0^2 A \cdot t \cdot A(t+\tau) \frac{1}{2} \, dA$$

$$= \frac{t}{2} \frac{(t+\tau)}{1} \int_0^2 A^2 \, dA$$

$$= \frac{t(t+\tau)}{2} \left. \frac{A^3}{3} \right|_0^2$$

$$= \frac{t(t+\tau)}{2} \times \frac{8}{3}$$

$$= \frac{4t(t+\tau)}{3}$$

$R_{xx}(t, \cancel{t+\tau}) = \frac{4}{3} t(t+\tau)$ is a function of time 't'.

Hence the given random process is <u>not a WSS random process</u> becoz two conditions are not satisfied.

2. A random process is described by $X(t) = A$ where A is a continuous random variable uniformly distributed on $(0,1)$ density. Find whether $X(t)$ is WSS random process or not.

Sol: Given random process $X(t) = A$

We know density function of uniformly distributed random variable X on (a,b) i.e.,

$$f_X(x) = \begin{cases} \frac{1}{b-a} & ; a \leq x \leq b \\ 0 & ; \text{elsewhere} \end{cases}$$

For the given problem A is the uniformly distributed on $(0,1)$ which is a random variable

$$f_A(A) = \begin{cases} \frac{1}{1} & ; 0 \leq A \leq 1 \\ 0 & ; \text{elsewhere} \end{cases}$$

We know condition for WSS process,

i) $E[X(t)] = \overline{X} =$ constant

ii) $R_{XX}[(t, t+\tau)] = E[X(t)\, X(t+\tau)] = R_{XX}(\tau)$ - is not a function of time t

i) The mean value of $X(t)$:

$$\overline{X} = \int_{-\infty}^{\infty} x(t)\, f_A(A)\, dA$$

$$= \int_{-\infty}^{0} (0)\, dA + \int_{0}^{1} A \cdot \frac{1}{1}\, dA + \int_{1}^{\infty} (0)\, dA$$

$$= \int_{0}^{1} A \cdot \frac{1}{1}\, dA$$

$$= \frac{1}{1} \cdot \left. \frac{A^2}{2} \right|_0^1$$

$$\overline{X} = \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{2} \Rightarrow \overline{X} = \frac{1}{2} = \text{constant}$$

(ii) Auto correlation of $X(t)$:

$R_{xx}(t, t+\tau) = E[X(t) \cdot X(t+\tau)] = R_{xx}(\tau)$

$$= \int_0^1 A \cdot A \, 1 \, dA$$

$$= \frac{A^3}{3}\Big|_0^1$$

$R_{xx}(\tau) = \frac{1}{3} = R_{xx}(t, t+\tau)$ = is not a function of time 't'.

Hence the given random process is a WSS random process becoz two conditions are satisfied.

3. Prove that the random process $x(t) = A\cos(\omega_c t + \theta)$ is WSS if it is assumed that W_c is a constant and θ is uniformly distributed random variable in the interval $(0, 2\pi)$.

Sol: We know the conditions for a WSS random process

(i) $E[x(t)] = \bar{X}$ = constant

(ii) $R_{xx}[(t, t+\tau)] = E[x(t)\,x(t+\tau)] = R_{xx}(\tau)$ is not a function of time 't'.

Given $x(t) = A\cos(\omega_c t + \theta)$

For the given random process $x(t)$, θ is uniformly distributed r.v. in the interval $(0, 2\pi)$.

$\therefore$ The PDF of $x(t) = f_\theta(\theta) = \begin{cases} \frac{1}{2\pi - 0} & 0 \le \theta \le 2\pi \\ 0 & \text{elsewhere} \end{cases}$

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & ; \; 0 \le \theta \le 2\pi \\ 0 & ; \; \text{elsewhere} \end{cases}$$

The mean value of random process $x(t)$: $E[x(t)] = \bar{X}$

$$= \int_{-\infty}^{\infty} x(t)\, f_\theta(\theta)\, d\theta$$

8. Prove that the random process

$$= \int_{-\infty}^{0} (0)\, d\theta + \int_{0}^{2\pi} \frac{1}{2\pi} A\cos(\omega_c t+\theta)\, d\theta + \int_{2\pi}^{\infty} (0)\, d\theta$$

$$= \frac{1}{2\pi}\int_{0}^{2\pi} A\cos(\omega_c t+\theta)\, d\theta$$

$$= \frac{A}{2\pi}\int_{0}^{2\pi} \cos(\omega_c t+\theta)\, d\theta$$

$$= \frac{A}{2\pi} \left. \frac{\sin(\omega_c t+\theta)}{cp} \right|_{0}^{2\pi}$$

$$= \frac{A}{2\pi}\left[\sin(\omega_c t+2\pi) - \sin(\omega_c t+0)\right]$$

$$= \frac{A}{2\pi}\left[\sin(\omega_c t) - \sin(\omega_c t)\right]$$

$$= \frac{A}{2\pi}(0) \qquad \{\because \sin(2\pi+\theta) = \sin\theta\}$$

$\therefore\ \overline{X} = 0 =$ constant

The autocorrelation of $x(t)$ is $= R_{xx}(t, t+\tau) = E[x(t)\, x(t+\tau)]$

$$= \int_{-\infty}^{\infty} x(t)\, x(t+\tau)\, f_\theta(\theta)\, d\theta$$

$$= \int_{0}^{2\pi} A\cos(\omega_c t+\theta) \cdot A\cos(\omega_c(t+\tau)+\theta) \cdot \frac{1}{2\pi}\, d\theta$$

$$= \frac{A^2}{2\pi}\int_{0}^{2\pi} \cos(\omega_c t+\theta)\cos(\omega_c(t+\tau)+\theta)\, d\theta$$

$$\because \cos A\cos B = \frac{\cos(A-B)+\cos(A+B)}{2}$$

$$= \frac{A^2}{2\pi}\cdot\frac{1}{2}\int_0 \cos\left(\cancel{\omega_c t} + \cancel{\theta}) - (\cancel{\omega_c}(\cancel{t}+\tau)+\cancel{\theta})\right) + \cos\left((\omega_c t+\theta)+(\omega_c t+\omega_c\tau+\theta)\right] d\theta$$

$$= \frac{A^2}{4\pi}\int_0^{2\pi}\left[\cos(\omega_c\tau) + \cos(2\omega_c t+\omega_c\tau+2\theta)\right] d\theta$$

$$= \frac{A^2}{4\pi}\int_0^{2\pi}\cos(\omega_c\tau)\, d\theta + \frac{A^2}{4\pi}\int_0^{2\pi}\cos(2\omega_c t+\omega_c\tau+2\theta)\, d\theta$$

$$= \frac{A^2}{4\pi}\cos(\omega_c\tau)\int_0^{2\pi} d\theta + \frac{A^2}{4\pi}\int_0^{2\pi}\cos(2(\omega_c t+\theta)+\omega_c\tau)\, d\theta$$

$$= \frac{A^2}{4\pi}\cos(\omega_c\tau)\left[\theta\right]_0^{2\pi} + \frac{A^2}{4\pi}\left[\frac{\sin(2(\omega_c t+\theta)+\omega_c\tau)}{2}\right]_0^{2\pi}$$

$$= \frac{A^2}{\cancel{4\pi}_{2}}\cos(\omega_c\tau)(\cancel{2\pi}) + \frac{A^2}{8\pi}\left(\sin(2\omega_c t+\omega_c\tau+4\pi) - \sin(2\omega_c t+\omega_c\tau)\right)$$

$$\{\because \sin(4\pi+\theta) = \sin\theta\}$$

$$= \frac{A^2}{2}\cos(\omega_c\tau) + \frac{A^2}{8\pi}\left(\sin(2\omega_c t+\omega_c\tau) - \sin(2\omega_c t+\omega_c\tau)\right)$$

$$= \frac{A^2}{2}\cos(\omega_c\tau) + \frac{A^2}{8\pi}(0)$$

$$\therefore R_{xx}(\tau) = \frac{A^2}{2}\cos(\omega_c\tau) = R_{xx}(t,(t+\tau)) \quad \text{(ii)}$$

Hence, (i) and (ii) are proved so the given random process is a WSS random process.

4. A random process is defined by $Y(t) = X(t)\cos(\omega_0 t + \theta)$ where $X(t)$ is an WSS random process that amplitude modulates a carrier of constant angular frequency ω_0, with a random phase θ independent of $X(t)$ and uniformly distributed on $(-\pi, \pi)$. Find

(i) $E[Y(t)]$

(ii) A.C.F of $Y(t)$

(iii) Is $Y(t)$ WSS or not?

Sol: Given random process $Y(t) = X(t)\cos(\omega_0 t + \theta)$ where $X(t)$ is WSS random process, i.e.,

(a) $E[X(t)] = \bar{X} = \text{constant}$

(b) $R_{XX}(t, t+\tau) = E[X(t)\,X(t+\tau)] = R_{XX}(\tau)$

ω_0 is constant and θ is uniformly distributed random r.v. on $(-\pi, \pi)$.

$\therefore$ The PDF of $Y(t) = f_\theta(\theta) = \begin{cases} \dfrac{1}{2\pi} & ; -\pi \le \theta \le \pi \\ 0 & ; \text{elsewhere} \end{cases}$

Conditions for WSS of $Y(t)$

(i) $E[Y(t)] = \cancel{\bar{Y} = \text{Constant}} = E[X(t)\cos(\omega_0 t + \theta)]$

$$E[Y(t)] = \int_{-\infty}^{\infty} Y(t)\, f_\theta(\theta)\, d\theta$$

$$= \int_{-\infty}^{\infty} X(t)\cos(\omega_0 t + \theta)\frac{1}{2\pi}\, d\theta$$

$$= \int_{-\pi}^{\pi} \frac{1}{2\pi}\, X(t)\cos(\omega_0 t + \theta)\, d\theta$$

$$= \frac{X(t)}{2\pi}\int_{-\pi}^{\pi} \cos(\omega_0 t + \theta)\, d\theta\Big] \times$$

(i) $E[Y(t)] = E[X(t)\cos(\omega_0 t+\theta)]$.

Given that θ and $X(t)$ are independent r.v's $\because E[XY]=E[X]E[Y]$

$$E[Y(t)] = E[X(t)]\; E[\cos(\omega_0 t+\theta)]$$

$$= E[X(t)] \int_{-\infty}^{\infty} \cos(\omega_0 t+\theta)\, f_\theta(\theta)\, d\theta$$

$$= E[X(t)] \int_{-\pi}^{\pi} \cos(\omega_0 t+\theta) \frac{1}{2\pi}\, d\theta$$

$$= \frac{E[X(t)]}{2\pi} \int_{-\pi}^{\pi} \cos(\omega_0 t+\theta)\, d\theta$$

$$= \frac{E[X(t)]}{2\pi} \left.\frac{\sin(\omega_0 t+\theta)}{1}\right|_{-\pi}^{\pi}$$

$$= \frac{E[X(t)]}{2\pi} \left[\frac{\sin(\omega_0 t+\pi) - \sin(\omega_0 t-\pi)}{1}\right]$$

$$= \frac{E[X(t)]}{2\pi} \left[\frac{\sin(\omega_0 t) + [\sin(\pi-\omega_0 t)]}{1}\right]$$

$$= \frac{E[X(t)]}{2\pi} [\sin(\omega_0 t) - \sin(\omega_0 t)]$$

$\because \sin(\pi+\theta) = \sin\theta$

$\sin(\pi-\theta) = -\sin\theta$

$$\boxed{\therefore\ \overline{Y} = 0 = \text{constant}}$$

(ii) Auto correlation func of $Y(t) = R_{YY}(t, t+\tau) = E[Y(t)\, Y(t+\tau)]$

$$= E[X(t)\cos(\omega_0 t+\theta)\; X(t+\tau)\cos(\omega_0 t + \omega_0\tau+\theta)]$$

$$= E[X(t)\, X(t+\tau) \cos(\omega_0 t+\omega_0\tau+\theta)\cos(\omega_0 t+\theta)]$$

Here $X(t)$ and θ are independent r.v's so

$$= E[X(t)\, X(t+\tau)] \cdot E[\cos(\omega_0 t+\omega_0\tau+\theta)\cos(\omega_0 t+\theta)]$$

$$= E[X(t)\, X(t+\tau)] \int_{-\infty}^{\infty} \cos(\omega_0 t+\omega_0\tau+\theta)\cos(\omega_0 t+\theta) \cdot \frac{1}{2\pi}\, d\theta$$

$$= \frac{E(X(t)\,X(t+\tau))}{2\pi} \int_{-\pi}^{\pi} \cos(\omega_0 t + \omega_0 \tau + \theta)\cos(\omega_0 t + \theta)\, d\theta$$

$\because \cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

$$= \frac{E[X(t)\,X(t+\tau)]}{2\pi \times 2} \int_{-\pi}^{\pi} [\cos(\omega_0 t + \omega_0 \tau + \theta - \omega_0 t - \theta) + \cos(\omega_0 t + \omega_0 \tau + \theta + \omega_0 t + \theta)]\, d\theta$$

$$= \frac{E[X(t)\,X(t+\tau)]}{4\pi} \int_{-\pi}^{\pi} \cos(\omega_0 \tau)\, d\theta + \cos(2\omega_0 t + \omega_0 \tau + 2\theta)\, d\theta$$

$$= \frac{1}{4\pi} E[X(t)\,X(t+\tau)] \cos(\omega_0 \tau) \int_{-\pi}^{\pi} d\theta$$

$$+ \frac{1}{4\pi} E[X(t)\,X(t+\tau)] \int_{-\pi}^{\pi} \cos(2\omega_0 t + \omega_0 \tau + 2\theta)\, d\theta$$

$$:= (0)$$

$$= \frac{1}{4\pi} E[X(t)\,X(t+\tau)] \cos \omega_0 \tau\, (2\pi) + 0$$

$$= \frac{1}{2} E[X(t)\,X(t+\tau)] = R_{YY}(t, t+\tau)$$

$R_{YY}(t, t+\tau) = \frac{1}{2} R_{XX}(\tau)$ which is a function of τ.

$$\boxed{\therefore \quad R_{YY}(t, t+\tau) = \frac{1}{2} R_{XX}(\tau)}$$

(iii) Conditions for WSS of $Y(t)$

(a) $E[Y(t)] = \bar{Y} =$ constant

(b) $R_{YY}(t, t+\tau) = E[Y(t)\,Y(t+\tau)] = R_{YY}(\tau)$.

$\therefore$ Two conditions are satisfied for the given random process $Y(t)$ hence, the given r.p is a WSS process

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* Time Averages of Random Processes:

> Time Average Function:

Consider a random process $X(t)$, let $x(t)$ be a sample function exists for all time at a fixed sample space S. The average value of $x(t)$ taken for all time is called time averages of random process $X(t)$ or the mean value of random process $X(t)$. It is denoted as the time average of $X(t)$ is equal to

$$\overline{x} = A[x(t)] \text{ or } \langle x(t) \rangle$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt.$$

> Time average of mean square value of $X(t)$:

Let the random process $X(t)$ with sample function $x(t)$ then the time average of $X^2(t)$ is called as time average of mean square value of $X(t)$, i.e.,

$$\overline{x^2} = A[x^2(t)] = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} x^2(t)\, dt$$

> Time average of auto-correlation function:

Let the random process $X(t)$, with sample function $x(t)$ then the time average of $x(t) \cdot x(t+\tau)$ is known as time auto-correlation function, i.e.,

$$\text{Time ACF of } X(t) = R_{xx}(\tau) = A[X(t)\, X(t+\tau)]$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\, dt$$

> Time Cross-Correlation function:

let two random processes $X(t)$ and $Y(t)$ with sample functions $x(t)$ and $y(t)$, respectively, then the time average of $x(t)\, y(t+\tau)$ is known as time cross-correlation function, i.e.,

$$\text{Time CCF of } X(t) \,\&\, Y(t) = R_{xy}(\tau)$$

$$= A[x(t)\, y(t+\tau)]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, y(t+\tau)\, dt$$

*Note:

All ensemble or statistical averages are not same as time averages.

$$A[\cdot] = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [\ ]\, dt$$

$$E[\cdot] = \int_{-\infty}^{\infty} [\cdot]\, f_X(x)\, dx$$

* Ergodic Theorem & (i) Ergodic Random Processes:

Statement:

The Ergodic theorem states that "for any random processes $X(t)$, all the time averages of sample function of $X(t)$ are equal to the corresponding Statistical or ensemble averages of $x(t)$" i.e.,

(i) Mean

$$\overline{x} = \overline{X}$$

Time average of $x(t)$ = Ensemble average of $X(t)$

$$A[x(t)] = E[X(t)]$$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [x(t)]\, dt = \int_{-\infty}^{\infty} [x(t)]\, f_X(x)\, dx$$

$$R_{XX}(\tau) = R_{XX}(t, t+\tau)$$

(ii) Time ACF of $X(t)$ = Ensemble ACF of $X(t)$

$$\underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x(t)\, x(t+\tau)\,dt = \int_{-\infty}^{\infty} x(t)\, x(t+\tau)\, f_X(x)\,dx$$

The random processes that satisfy Ergodic theorem are called Ergodic random process.

(ii) * Jointly Ergodic Random Processes:

Let two random processes $X(t)$ and $Y(t)$ with samples functions $x(t)$ and $y(t)$, respectively. The two random processes are said to be jointly ergodic, if they are individually ergodic and also the time cross-correlation function is equal to the ensemble cross-correlation function of $x(t)$ and $y(t)$ i.e.,

(i) $\bar{x} = \bar{X}$ & $\bar{y} = \bar{Y}$

$A[x(t)] = E[X(t)]$ $\quad$ $A[y(t)] = E[Y(t)]$

(ii) $R_{xx}(\tau) = R_{XX}(t, t+\tau)$ & $R_{yy}(\tau) = R_{YY}(t, t+\tau)$

$A[x(t)\, x(t+\tau)] = E[X(t)\, X(t+\tau)]$ $\quad$ $A[y(t)\, y(t+\tau)] = E[Y(t)\, Y(t+\tau)]$

(iii) CCF

$$R_{xy}(\tau) = R_{XY}(t, t+\tau)$$

$$A[x(t)\, y(t+\tau)] = E[X(t)\, Y(t+\tau)]$$

Here $A[\cdot] = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} [\cdot]\, dt$ & $E[\cdot] = \int_{-\infty}^{\infty} [\cdot]\, f_X(x)\, dx$

Time Average $\qquad$ Ensemble Average

* (iii) Mean Ergodic Random Process:

Let the random process $X(t)$ with sample function $x(t)$ is said to be mean ergodic random process if and only if its time average is equal to the ensemble average of $X(t)$, i.e.,

$$\bar{x} = \bar{X}$$

Time average of $x(t)$ = Ensemble average of $X(t)$

$$A[x(t)] = E[X(t)]$$

$$\underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T}[x(t)]\,dt = \int_{-\infty}^{\infty}[X(t)]\,f_X(x)\,dx$$

* (iv) ACF Ergodic Random Process:

Let the random process $X(t)$ with sample function $x(t)$ is said to be auto correlation function ergodic random process if and only if its time autocorrelation function is equal to the its ensemble auto correlation function of $X(t)$

$$R_{xx}(\tau) = R_{XX}(t, t+\tau)$$

$$\underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x(t)\,x(t+\tau)\,dt = \int_{-\infty}^{\infty} x(t)\,x(t+\tau) f_X(x)\,dx$$

Time average ACF of $x(t)$ = Ensemble ACF of $X(t)$

* (v) Cross-Correlation Function Ergodic Random Process:

Let the random processes $X(t)$ and $Y(t)$ with sample functions $x(t)$ and $y(t)$, respectively are said to be CCFERP if and only if ~~and only~~ if time CCF of $x(t)$ & $y(t)$ are equal to ensemble CCF of $X(t)$ & $Y(t)$.

$$R_{xy}(\tau) = R_{xy}(t, t+\tau)$$

$$A[X(t)\ Y(t+\tau)] = E[X(t)\ Y(t+\tau)]$$

$$\underset{T\to\infty}{Lt}\ \frac{1}{2T}\int_{-T}^{T} X(t)\ Y(t+\tau)\,dt = \int_{-\infty}^{\infty} X(t)\ Y(t+\tau)\ f_X(x)\,dx$$

5. Let the random process $X(t) = A\cos(\omega_0 t + \theta)$ if it is assumed that ω_0 is a constant and θ is uniformly distributed r. variable on the interval $(-\pi, \pi)$.

(i) Verify $X(t)$ is WSS or not.

(ii) Is $x(t)$ is ergodic random process or not.

Sol: Given r.p. $X(t) = A\cos(\omega_0 t + \theta)$, where ω_0 is constant and θ is uniformly distributed r.v on $(-\pi, \pi)$

We know that PDF of $\theta = f_\theta(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi \le \theta \le \pi \\ 0, & \text{elsewhere} \end{cases}$

* Conditions for WSS r.p.

(i) $E[X(t)] = \bar{X} =$ constant

$$\bar{X} = \int_{-\infty}^{\infty} x(t)\ f_\theta(\theta)\,d\theta$$

$$= \int_{-\infty}^{-\pi} (0)\,dA + \int_{-\pi}^{\pi} \frac{1}{2\pi} A\cos(\omega_0 t + \theta)\,d\theta + \int_{\pi}^{\infty} 0\,d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} A\cos(\omega_0 t + \theta)\,d\theta$$

$$= \frac{A}{2\pi}\ \frac{\sin(\omega_0 t + \theta)}{1}\Bigg|_{2\pi}^{2\pi}$$

$$= \frac{A}{2\pi}\left[(\sin(\omega_0 t + \pi) - \sin(\omega_0 t - \pi)\right]$$

$$= \frac{A}{2\pi}\left[\sin(\omega_0 t) - \sin(\omega_0 t)\right]$$

$$= \frac{A}{2\pi}[0]$$

$$\overline{X} = 0 = \text{constant}$$

(ii) The autocorrelation of $x(t)$ is $R_{xx}(t, t+\tau) = E[x(t)\,x(t+\tau)]$

$$= \int_{-\pi}^{\pi} x(t)\,x(t+\tau)\,f_\theta(\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} A\cos(\omega_0 t+\theta) * A\cos(\omega_0(t+\tau)+\theta)\,d\theta$$

$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \cos(\omega_0 t+\theta)\cos(\omega_0 t+\omega_0\tau+\theta)\,d\theta$$

$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \frac{\cos(\omega_0 t+\theta+\omega_0 t+\omega_0\tau+\theta)+\cos(\cancel{\omega_0 t}+\cancel{\theta}-\cancel{\omega_0 t}+\omega_0\tau-\cancel{\theta})}{2}$$

$$= \frac{A^2}{2\pi\times 2}\int_{-\pi}^{\pi} \cos(2\omega_0 t+\omega_0\tau+2\theta) + \cos(+\omega_0\tau)\,d\theta$$

$$= \frac{A^2}{4\pi}\int_{-\pi}^{\pi} \frac{\cos(2(\omega_0 t+\theta)+\omega_0\tau)+\cos(\omega_c\tau)}{1}\,d\theta$$

$$= \frac{A^2}{4\pi}\int_{-\pi}^{\pi} \frac{\cos(2(\omega_0 t+\theta)+\omega_0\tau)}{1} + \frac{A^2}{4\pi}\int_{-\pi}^{\pi} \cos(\omega_c\tau)\,d\theta$$

$$= \frac{A^2}{4\pi}\cos(\omega_0\tau)\,[\theta]_{-\pi}^{\pi} + \frac{A^2}{4\pi}\left.\frac{\sin(2(\omega_0 t+\theta)+\omega_0\tau)}{2}\right|_{-\pi}^{\pi}$$

$$= \frac{A^2}{4\pi}\cos\omega_0\tau\,[\pi+\pi] + \frac{A^2}{8\pi}\left[\sin(2(\omega_0 t+\pi)+\omega_0\tau) - \sin 2(\omega_0 t-\pi)+\omega_0\tau\right]$$

$$= \frac{A^2}{4\pi} \cos \omega_0 \tau (2\pi) + \frac{A^2}{8\pi}(0)$$

$$= \frac{A^2}{\cancel{4\pi}2} \cos \omega_0 \tau (\cancel{2\pi})$$

$$R_{xx}(t, t+\tau) = \frac{A^2}{2} \cos \omega_0 \tau = R_{xx}(\tau)$$

Hence the mean of $x(t)$ is constant and ACF of $x(t)$ is a function of τ so <u>the given process is a WSS process.</u>

(ii) Conditions for ergodic random process are:

(a) Time average of $x(t)$ = Ensemble average of $x(t)$

$$\bar{x} = \bar{X}$$

$$A_T[x(t)] = E[X(t)]$$

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt = \int_{-\infty}^{\infty} x(t) f_\theta(\theta)\, d\theta$$

(b) $$R_{xx}(\tau) = R_{xx}(t, t+\tau)$$

$$A[x(t)\, x(t+\tau)] = E[x(t)\, x(t+\tau)]$$

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\, dt = \int_{-\infty}^{\infty} x(t)\, x(t+\tau) f_\theta(\theta)\, d\theta$$

* Ensemble Averages:

Ensemble mean of $x(t) = \bar{X} = E[x(t)] = \int_{-\infty}^{\infty} x(t) f_\theta(\theta)\, d\theta$

$$\boxed{E[x(t)] = 0 = \text{constant}}$$

Ensemble ACF of $x(t) = E[x(t)\, x(t+\tau)] = \int_{-\infty}^{\infty} x(t)\, x(t+\tau) f_\theta(\theta)\, d\theta$

$\boxed{R_{xx}(t, t+\tau) = \frac{A^2}{2} \cos(\omega_0 \tau)}$ which is a function of 'τ' only and not absolute time 't'.

* Time Averages:

Time average of $x(t) = \bar{x} = A[x(t)] = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x(t)\, dt$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} A\cos(\omega_0 t+\theta)\, dt \qquad \boxed{\omega_0 = \frac{2\pi}{T}\text{ rad/sec}}$$

$$= \underset{T\to\infty}{Lt} \frac{A}{2T} \left.\frac{\sin(\omega_0 t+\theta)}{\omega_0}\right|_{-T}^{T}$$

$$= \underset{T\to\infty}{Lt} \frac{A}{2\cancel{T}} \frac{\sin\left(\frac{2\pi}{T}(T)+\theta\right) - \sin\left(\frac{2\pi}{T}(-T)+\theta\right)}{\frac{2\pi}{\cancel{T}}}$$

$$= \underset{T\to\infty}{Lt} \frac{A}{4\cancel{T}} \frac{\sin(2\pi+\theta) - \sin(-2\pi+\theta)}{\pi}$$

$$= \underset{T\to\infty}{Lt} \frac{A}{4\pi} \frac{\sin(2\pi+\theta) - (-\sin(2\pi-\theta))}{1}$$

$$= \underset{T\to\infty}{Lt} \frac{A}{4\pi} [\cancel{\sin}\theta - \cancel{\sin}\theta]$$

$$= \underset{T\to\infty}{Lt} \frac{A}{4\pi} (0)$$

$$\therefore \bar{x} = 0 = A[x(t)]$$

Time ACF of $x(t) = A[x(t)\, x(t+\tau)]$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x(t)\, x(t+\tau)\, dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} A\cos(\omega_0 t+\theta) \cdot A\cos(\omega_0(t+\tau)+\theta)\, dt$$

$$= \underset{T\to\infty}{Lt} \frac{A^2}{2T}\int_{-T}^{T} \cos(\omega_0 t+\theta)\cos(\omega_0 t+\omega_0\tau+\theta)\, dt$$

$$= \underset{T\to\infty}{Lt} \frac{A^2}{2T}\int_{-T}^{T} \frac{\cos(\omega_0 t \cancel{+\theta} - (\omega_0 t+\omega_0\tau+\theta)) + \cos(2\omega_0 t+\omega_0\tau+2\theta)}{2}$$

$$= \underset{T\to\infty}{Lt} \frac{A^2}{4T} \int_{-T}^{T} \cos(\omega_0\tau)\,dt + \cos(2(\omega_0 t+\theta)+\omega_0\tau)\,dt$$

$$= \underset{T\to\infty}{Lt} \frac{A^2}{4T} \int_{-T}^{T} \cos(\omega_0\tau)\,dt + \underset{T\to\infty}{Lt} \frac{A^2}{4T} \int_{-T}^{T} \cos 2(\omega_0 t+\theta)+\omega_0\tau\,dt$$

$$= \underset{T\to\infty}{Lt} \frac{A^2\cos(\omega_0\tau)}{4T}(t)_{-T}^{T} + \underset{T\to\infty}{Lt} \frac{A^2}{4T} \left.\frac{\sin 2(\omega_0 t+\theta)+\omega_0\tau}{2\omega_0}\right|_{-T}^{T}$$

$$= \underset{T\to\infty}{Lt} \frac{A^2}{\cancel{4T}}\cos(\omega_0\tau)(\cancel{2T}) + \underset{T\to\infty}{Lt} \frac{A^2}{4\cancel{T}\cdot 2\cdot\frac{2\pi}{\cancel{T}}} \left[\frac{\sin\left(2\left(\frac{2\pi}{T}t+\theta\right)+\omega_0\tau + \frac{2\pi}{T}(\tau)\right)}{1}\right]_{-T}$$

$$= \underset{T\to\infty}{Lt} \frac{A^2}{2}\cos(\omega_0\tau) + \underset{T\to\infty}{Lt} \frac{A^2}{8\pi}\left[\sin(2(2\pi+\theta)+\omega_0\tau) - \sin 2\,\frac{2\pi}{T}(-T)+\omega_0\tau\right]$$

$$= \underset{T\to\infty}{Lt} \frac{A^2}{2}\cos\omega_0(\tau) + \underset{T\to\infty}{Lt} \frac{A^2}{8\pi}\left[\sin(4\pi+\omega_0\tau+2\theta) - \sin(-4\pi+\omega_0\tau+2\theta)\right]$$

$$= \underset{T\to\infty}{Lt} \frac{A^2}{2}\cos\omega_0(\tau) + \underset{T\to\infty}{Lt} \frac{A^2}{8\pi}\left[\sin[4\pi+(\omega_0\tau+2\theta)] - (-\sin(4\pi-(\omega_0\tau+2\theta)))\right]$$

$$= \underset{T\to\infty}{Lt} \frac{A^2}{2}\cos\omega_0\tau + \underset{T\to\infty}{Lt} \frac{A^2}{8\pi}[0]$$

$$= \underset{T\to\infty}{Lt} \frac{A^2}{2}\cos\omega_0\tau + 0$$

$$= \underset{T\to\infty}{Lt} \frac{A^2}{2}\cos\omega_0\tau \qquad \because T \text{ is not present in the term.}$$

$$\boxed{R_{xx}(t,t+\tau) = \frac{A^2}{2}\cos\omega_0\tau}$$

$\therefore$ (i) $A[x(t)] = E[x(t)] \Rightarrow \bar{x} = \bar{X}$

(ii) $R_{xx}(\tau) = R_{xx}(t,t+\tau) \Rightarrow A[x(t)\,x(t+\tau)]$
$= E[x(t)\,x(t+\tau)]$

Hence time averages of $x(t)$ = Ensemble averages of $x(t)$ so it is an ergodic random process.

6. Two random processes are defined by $X(t) = A\cos(\omega_0 t + \theta)$, $Y(t) = B\sin(\omega_0 t + \theta)$ where θ is an uniform r.v on $(0, 2\pi)$ and A, B and ω_0 are constants. Find out

(i) Verify whether $X(t)$ and $Y(t)$ are jointly WSS or not

(ii) Verify whether $X(t)$ and $Y(t)$ are ergodic random process or not.

Sol: Given random processes $X(t) = A\cos(\omega_0 t + \theta)$

$Y(t) = B\sin(\omega_0 t + \theta)$ where

A, B and ω_0 are constants.

θ is an uniform r.v on $(0, 2\pi)$ so we have

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi}; & 0 \le \theta \le 2\pi \\ 0; & \text{elsewhere} \end{cases}$$

$f_\theta(\theta)$ is the PDF of the given function

i) Conditions for WSS r.p:

(a) The mean value of $x(t) = \overline{X} = E[x(t)]$

$$= \int_{-\infty}^{\infty} x(t)\, f_\theta(\theta)\, d\theta$$

$$= \int_{-\infty}^{0} (0)\, d\theta + \int_{0}^{2\pi} A\cos(\omega_0 t + \theta)\, \frac{1}{2\pi}\, d\theta + \int_{2\pi}^{\infty} (0)\, d\theta$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} A\cos(\omega_0 t + \theta)\, d\theta$$

$$= \frac{A}{2\pi} \int_{0}^{2\pi} \cos(\omega_0 t + \theta)\, d\theta$$

$$= \frac{A}{2\pi} \left[\frac{\sin(\omega_0 t + \theta)}{\omega_0}\right]_0^{2\pi}$$

$\overline{X} = 0 =$ constant

The mean value of $Y = \overline{Y} = E[Y(t)]$

$R_{XY}(t, t+\tau) = E[X(t)\, Y(t+\tau)] = \int_{-\infty}^{\infty} x(t)\, y(t+\tau) f_\theta(\theta)\, d\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} A\sin(\omega_0 t+\theta)\, d\theta$$

$$= \frac{A}{2\pi} \; +\cos(\omega_0 t+\theta)\Big|_0^{2\pi}$$

$$= \frac{A}{2\pi}(0)$$

$$\overline{Y} = 0 = \text{constant}$$

The cross correlation of $X(t)$ and $Y(t)$ is given as

$$= R_{XY}(t, t+\tau) = E[X(t)\; Y(t+\tau)]$$

$$= \int_{-\infty}^{\infty} x(t)\, y(t+\tau)\, f_\theta(\theta)\, d\theta$$

$$= \int_{-\infty}^{0} (0) f_\theta(\theta)\, d\theta + \int_0^{2\pi} A\cos(\omega_0 t+\theta)\, B\sin(\omega_0(t+\tau)+\theta)\cdot\frac{1}{2\pi}\, d\theta + \int_{2\pi}^{\infty} (0)\, d\theta$$

$$= \int_0^{2\pi} \frac{AB}{2\pi} \cos(\omega_0 t+\theta)\sin(\omega_0 t+\omega_0\tau+\theta)\, d\theta$$

$$= \frac{AB}{2\pi}\int_0^{2\pi} \cos(\omega_0 t+\theta)\sin(\omega_0 t+\omega_0\tau+\theta)\, d\theta$$

$$= \frac{AB}{2\pi}\int_0^{2\pi} \frac{\sin(\omega_0 t+\theta+\omega_0 t+\omega_0\tau+\theta) - \sin(\omega_0 t+\theta-\omega_0 t-\omega_0\tau-\theta)}{2}\, d\theta$$

$$= \frac{AB}{2\pi\times 2}\int_0^{2\pi} \underbrace{\sin(2\omega_0 t+\omega_0\tau+2\theta)}_{\text{(This term becomes zero)}} + \sin(\omega_0\tau)\, d\theta$$

$$= \frac{AB}{4\pi}\int_0^{2\pi} \sin(\omega_0\tau)\, d\theta$$

$$= \frac{AB}{4\pi} \sin(\omega_0 \tau) \, [\theta]_0^{2\pi}$$

$$= \frac{AB}{2} \sin(\omega_0 \tau)$$

(ii) Conditions for ergodic random process

(a) Time average of $x(t)$ = Ensemble average of $x(t)$

$$\bar{x} = \bar{X}$$

$$A[X(t)] = E[x(t)]$$

$$\underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt = \int_{-\infty}^{\infty} x(t)\, f_\theta(\theta)\, d\theta$$

(b) Cross correlation function of $x(t)$

Time average of CCF of $x(t)$ = Ensemble average of CCF of $x(t)$

$$R_{xy}(\tau) = R_{xy}(t, t+\tau)$$

$$A[x(t)\, Y(t+\tau)] = E[x(t)\, x(t+\tau)]$$

$$\underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} x(t)\, Y(t+\tau)\, dt = \int_{-\infty}^{\infty} x(t)\, y(t+\tau)\, f_\theta(\theta)\, d\theta$$

* Ensemble averages:

Ensemble mean of $x(t) = \bar{X} = E[X(t)] = 0$ = constant

Ensemble mean of $Y(t) = \bar{Y} = E[Y(t)] = 0$ = constant

Ensemble average of CCF of $x(t)$ and $Y(t)$ is

$$R_{xy}(t, t+\tau) = E[X(t)\, Y(t+\tau)]$$

$$R_{xy}(t, t+\tau) = \frac{AB}{2} \sin \omega_0 \tau$$

* Time averages:

Time average of $x(t) = \bar{x} = A[x(t)] = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$

$$= \underset{T\to\infty}{Lt} \frac{A}{2T}\int_{-T}^{T} \cos(\omega_0 t+\theta)\,dt$$

$$= \underset{T\to\infty}{Lt} \frac{A}{2T}\left[\frac{\sin(\omega_0 t+\theta)}{\omega_0}\right]_{-T}^{T}$$

$$= \underset{T\to\infty}{Lt} \frac{A}{2T\cdot\frac{2\pi}{T}}\left[\frac{\sin\left(\frac{2\pi}{T}(T)+\theta\right)-\sin\left(\frac{2\pi}{T}(-T)+\theta\right)}{1}\right]$$

$$= \underset{T\to\infty}{Lt} \frac{A}{4\pi}\left[\frac{\sin(2\pi+\theta)-\sin(2\pi-\theta)}{1}\right]$$

$$= \underset{T\to\infty}{Lt} \frac{A}{4\pi}[0]$$

$\bar{x} = 0 =$ constant

Time average of $y(t) = \bar{Y} = A[y(t)] = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} y(t)\,dt$

$$= \underset{T\to\infty}{Lt} \frac{B}{2T}\int_{-T}^{T} \sin(\omega_0 t+\theta)\,dt$$

$$= \underset{T\to\infty}{Lt} \frac{B}{2T}\int_{-T}^{T} \sin(\omega_0 t+\theta)\,dt$$

$$= \underset{T\to\infty}{Lt} \frac{B}{2T}\cdot\frac{\cos(\omega_0 t+\theta)}{\omega_0}\Big|_{-T}^{T}$$

$$= \underset{T\to\infty}{Lt} \frac{B}{2T}\left[\frac{\cos\left(\frac{2\pi}{T}\cdot T+\theta\right)-\cos\left(\frac{2\pi}{T}(-T)-\theta\right)}{\frac{2\pi}{T}}\right]$$

$$= \underset{T\to\infty}{Lt} \frac{B}{2T}(0)$$

$\bar{Y} = 0 =$ constant.

Time average of CCF of $x(t)$ and $y(t)$ we have

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x(t)\,y(t+\tau)\,dt$$

$$= \lim_{T\to\infty} \frac{AB}{2T} \int_{-T}^{T} \cos(\omega_0 t+\theta)\sin(\omega_0(t+\tau)+\theta)\, dt$$

$$= \lim_{T\to\infty} \frac{AB}{2T\times 2} \int_{-T}^{T} \{\sin(\omega_0 t+\theta+\omega_0 t+\omega_0\tau+\theta) - \sin(\omega_0 t+\theta-\omega_0 t-\omega_0\tau-\theta)\}\, dt$$

$$= \lim_{T\to\infty} \frac{AB}{4T} \int_{-T}^{T} \sin(2(\omega_0 t+\theta)+\omega_0\tau)\, dt + \sin(\omega_0\tau)\, dt$$

$$= \lim_{T\to\infty} \frac{AB}{4T} \int_{-T}^{T} \sin(2(\omega_0 t+\theta)+\omega_0\tau)\, dt + \lim_{T\to\infty} \frac{AB}{4T} \int_{-T}^{T} \sin(\omega_0\tau)\, dt$$

$$= \lim_{T\to\infty} \frac{AB}{4T} \left. \frac{\cos(2(\omega_0 t+\theta)+\omega_0\tau)}{2\omega_0} \right|_{-T}^{T} + \lim_{T\to\infty} \frac{AB}{4T} \sin\omega_0\tau \int_{-T}^{T} dt$$

$$= 0 + \lim_{T\to\infty} \frac{AB}{4T} \sin(\omega_0\tau) [t]_{-T}^{T}$$

$$= \lim_{T\to\infty} \frac{AB}{\cancel{4T}_2} \sin(\omega_0\tau) [\cancel{2T}]$$

$$= \frac{AB}{2} \sin(\omega_0\tau)$$

Hence the (a) and (b) conditions for ergodic random process are satisfied, so the given two random processes $X(t)$ and $Y(t)$ are ergodic random processes.

* Properties of Auto Correlation Function:

v.v.imp Let us consider a random process $X(t)$ is atleast a WSS, then the following are properties of ACF.

1) The value of ACF of random process $x(t)$ at origin will gives mean square value of the random process $X(t)$ i.e., Average power of $x(t)$

$$\boxed{R_{XX}(0) = \overline{X^2} = E[X^2]}$$

Proof: The autocorrelation function of $x(t)$ is

$$= R_{XX}(t_1, t_2) = E[X(t_1)\, X(t_2)].$$

Given $x(t)$ is a WSS process

$$\tau = t_2 - t_1 \;,\; t_1 = t$$

$$\Rightarrow t_2 = \tau + t_1 = \tau + t$$

$$R_{XX}(t, t+\tau) = E[X(t)\, X(t+\tau)]$$

For WSS of $x(t)$, $R_{XX}(t, t+\tau) = R_{XX}(\tau)$

$$R_{XX}(\tau) = E[X(t)\, X(t+\tau)]$$

At origin, i.e. $\tau = 0$

$$R_{XX}(0) = E[X(t)\, X(t+0)]$$

$$R_{XX}(0) = E[X(t)\, X(t)]$$

$$R_{XX}(0) = E[X^2(t)]$$

$$\therefore R_{XX}(0) = \overline{X^2(t)} = \overline{X^2}$$

Hence proved.

2) The maximum value of ACF of $X(t)$ occurs at origin i.e $\boxed{|R_{XX}(\tau)| \leq R_{XX}(0)}$

Proof: We know the ACF of $X(t)$ is

$$= R_{XX}(t_1, t_2) = E[X(t_1)\, X(t_2)]$$

Consider a +ve random process i.e., $[X(t_1) \pm X(t_2)]^2 \geq 0$

Apply Expectation both sides we get,

$$E[X(t_1) \pm \cancel{E} X(t_2)]^2 \geq E[0]$$

$$E[X^2(t_1) + X^2(t_2) \pm 2X(t_1)X(t_2)] \geq 0$$

$\because E[0] = 0$

$E[X+Y] = E[X] + E[Y]$

$$E[X^2(t_1)] + E[X^2(t_2)] \pm 2E[X(t_1)X(t_2)] \geq 0 \rightarrow (1)$$

We know $R_{XX}(0) = \overline{X^2(t)} = E[X^2(t_1)] = E[X^2(t_2)] = E[X^2(t)]$

Now (1) $\Rightarrow$ $R_{XX}(0) + R_{XX}(0) \pm 2E[X(t_1)X(t_2)] \geq 0$

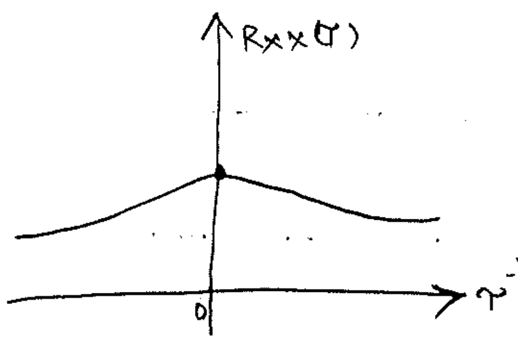
$$R_{XX}(0) + R_{XX}(0) \pm 2R_{XX}(t_1, t_2) \geq 0$$

Given $X(t)$ is a WSS random process, then

$$R_{XX}(t_1, t_2) = R_{XX}(t, t+\tau) = R_{XX}(\tau)$$

(1) $\Rightarrow$ $R_{XX}(0) + R_{XX}(0) \pm 2R_{XX}(\tau) \geq 0.$

$$\cancel{2} R_{XX}(0) \geq \mp \cancel{2} R_{XX}(\tau)$$



$$R_{XX}(0) \geq \pm R_{XX}(\tau)$$

$$R_{XX}(0) \geq |R_{XX}(\tau)|$$

$$\therefore R_{XX}(\tau) \leq R_{XX}(0)$$

Hence proved.

This means that as τ increases in either direction ACF will decreases, and τ decreases in either ~~direction~~ direction as ... and max. value of ACF occurs at origin.

3) The ACF of $X(t)$ satisfies even symmetry, i.e.,

$$\boxed{R_{xx}(\tau) = R_{xx}(-\tau) \quad \forall \tau}$$

Proof: We know ACF of $X(t)$ is $= R_{xx}(t_1, t_2) = E[X(t_1)X(t_2)]$

Given $X(t)$ is WSS process, then

$$R_{xx}(t_1,t_2) = R_{xx}(t, t+\tau) = R_{xx}(\tau)$$

$$R_{xx}(\tau) = E[X(t)\, X(t+\tau)]$$

Replace τ by $-\tau$, we get

$$R_{xx}(-\tau) = E[X(t)\, X(t+(-\tau))]$$

$$R_{xx}(-\tau) = E[X(t)\, X(t-\tau)]$$

Put $t-\tau = k$

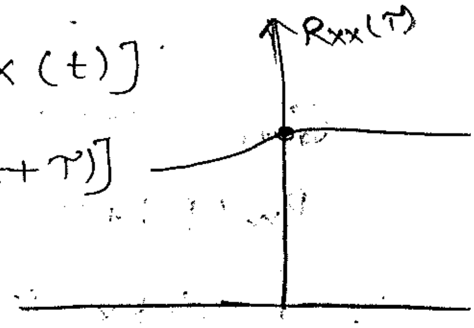
$$t-k = \tau \Rightarrow t = k+\tau$$

$$\Rightarrow R_{xx}(-\tau) = E[X(k+\tau)\, X(k)]$$

Put $k = t$

$$\Rightarrow R_{xx}(-\tau) = E[X(t+\tau)\, X(t)]$$

$$= E[X(t)\, X(t+\tau)]$$



$$\therefore R_{xx}(-\tau) = R_{xx}(\tau)$$

Hence proved.

4) Let the random process $X(t)$ has a non-zero mean value; $E[X(t)] = \overline{X} \neq 0$, and it is ergodic with no periodic components, then $\boxed{\lim_{|\tau| \to \infty} R_{xx}(\tau) = \overline{X}^2 = [E[X(t)]]^2}$

Proof: The ACF of random process $X(t) = R_{xx}(t_1, t_2)$

$$= E[X(t_1)\, X(t_2)]$$

Given $X(t)$ is a WSS and ergodic, then

$$R_{xx}(t, t+\tau) = E[X(t)\, X(t+\tau)] = R_{xx}(\tau)$$

$\therefore R_{xx}(\tau) = E[X(t_1)\, X(t_2)]$

As $\tau \to \infty$ the random variables $X(t_1)$ and $X(t_2)$ are independent, then

$$\underset{|\tau| \to \infty}{Lt} R_{xx}(\tau) = E[X(t_1)\, X(t_2)] = E[X(t_1)]\; E[X(t_2)]$$

For WSS random process,

$$E[X(t)] = E[X(t_1)] = E[X(t_2)] = \overline{X}.$$

$$\underset{|\tau| \to \infty}{Lt} R_{xx}(\tau) = \overline{X} \cdot \overline{X}$$

$$\underset{|\tau| \to \infty}{Lt} R_{xx}(\tau) = \overline{X^2} = \left[E[X(t)]\right]^2$$

Hence proved.

5) let the random process $X(t)$ has zero mean value and it is ergodic then $\boxed{\underset{|\tau| \to \infty}{Lt} R_{xx}(\tau) = 0}$

Proof: The ACF of random process $X(t) = R_{xx}(t_1, t_2)$

$$= E[X(t_1)\, X(t_2)]$$

Given $X(t)$ is WSS and ergodic, then

$$R_{xx}(t, t+\tau) = E[X(t)\, X(t+\tau)] = R_{xx}(\tau)$$

$$\therefore R_{xx}(\tau) = E[X(t_1)\, X(t_2)]$$

As $T \to \infty$ the random variables $X(t_1)$ and $X(t_2)$ are independent, then

$$\underset{T \to \infty}{Lt} R_{xx}(\tau) = E[X(t_1)\, X(t_2)] = E[X(t_1)]\; E[X(t_2)]$$

For WSS process,

$$E[X(t)] = E[X(t_1)] = E[X(t_2)] = \overline{X}$$

$$\underset{|\tau| \to \infty}{Lt} R_{xx}(\tau) = \overline{X} \cdot \overline{X}$$

$$\underset{|\tau| \to \infty}{Lt} R_{xx}(\tau) = \overline{X^2} = \left[E[X(t)]\right]^2$$

Given: $x(t)$ having zero mean value

i.e. $\overline{X} = E[x(t)] = 0$

$$\therefore \underset{|\tau| \to \infty}{Lt} R_{xx}(\tau) = \overline{X}^2 = (0)^2 = 0$$

Hence the theorem is proved.

6) If $x(t)$ is periodic then ACF is also periodic function

$$\boxed{R_{xx}(\tau \pm T_0) = R_{xx}(\tau)}$$

where T_0 is fundamental period of $x(t)$ and $R_{xx}(\tau)$.

Proof: The given r.p $X(t)$ is a periodic function i.e.

$$X(t) = X(t \pm T_0) \quad \forall t \longrightarrow (1)$$

$$\Rightarrow X(t+\tau) = X((t+\tau) \pm T_0)$$

The ACF of WSS of $X(t) = R_{xx}(\tau) = E[X(t) X(t+\tau)]$

$$R_{xx}(\tau \pm T_0) = E[X(t) X[t + (\tau \pm T_0)]]$$

$$= E[X(t) X(t + \tau \pm T_0)]$$

$$= E[X(t) X(t+\tau)] \quad \because \text{From (1)}$$

$$= R_{xx}(\tau)$$

7) Let $Z = X + Y$ then $R_{zz}(\tau) = R_{xx}(\tau) + R_{yy}(\tau) + R_{xy}(\tau) + R_{yx}(\tau)$.

Proof: Given $Z = X + Y \Rightarrow Z(t) = X(t) + Y(t)$

The ACF of r.p $Z(t) = R_{zz}(\tau) = E[Z(t) Z(t+\tau)]$

$$= E[(X(t) + Y(t)) \cdot (X(t+\tau) + Y(t+\tau))]$$

$$= E[(x(t)+y(t))(x(t+\tau)+y(t+\tau))]$$

$$= E[x(t)\,x(t+\tau) + x(t)\,y(t+\tau) + y(t)\,x(t+\tau) + y(t)\,y(t+\tau)]$$

$$= E[x(t)\,x(t+\tau)] + E[x(t)\,y(t+\tau)] + E[y(t)\,x(t+\tau)] + E[y(t)\,y(t+\tau)]$$

$$= R_{xx}(\tau) + R_{xy}(\tau) + R_{yx}(\tau) + R_{yy}(\tau)$$

$$\boxed{R_{zz}(\tau) = R_{xx}(\tau) + R_{yy}(\tau) + R_{xy}(\tau) + R_{yx}(\tau)}$$

Hence proved.

* Properties of Cross Correlation Function of Random Process:

1) Let two random processes $x(t)$ and $y(t)$ are atleast jointly WSS, then the following are properties of CCF

1) Symmetry Property:

$$\boxed{R_{xy}(\tau) = R_{yx}(-\tau)}$$

Proof: The cross correlation function of $x(t)$ and $y(t)$

$$= R_{xy}(\tau) = E[x(t)\,y(t+\tau)]$$

$$R_{yx}(\tau) = E[y(t)\,x(t+\tau)]$$

Replace τ by $-\tau$, we get

$$R_{yx}(-\tau) = E[y(t)\,x(t+(-\tau))]$$

Put $k = t-\tau$

$t = k+\tau$

$$= E[y(k+\tau)\,x(k)]$$

$$R_{YX}(-\tau) = E[Y(t+\tau)\ X(t)]$$
$$= E[X(t)\ Y(t+\tau)]$$
$$\therefore R_{YX}(-\tau) = R_{XY}(\tau)$$

Hence proved.

2) $|R_{XY}(\tau)| \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$

Proof: Consider a +ve random process i.e.,

$$[X(t) \pm Y(t+\tau)]^2 \geq 0$$

Applying expectation on both sides, we get

$$E[X(t) \pm Y(t+\tau)]^2 \geq E[0]$$

$$E[X^2(t) + Y^2(t+\tau)^2 \pm 2X(t)\ Y(t+\tau)] \geq 0$$

$$E[X^2(t)] + E[Y^2(t+\tau)] \pm 2E[X(t)\ Y(t+\tau)] \geq 0$$

We know $R_{XX}(0) = E[X^2(t)]$

$$R_{YY}(0) = E[Y^2(t)] = E[Y^2(t+\tau)]$$

$$R_{XX}(0) + R_{YY}(0) \pm 2R_{XY}(\tau) \geq 0$$

$$R_{XX}(0) + R_{YY}(0) \geq \mp 2R_{XY}(\tau)$$

$$\frac{1}{2}\ R_{XX}(0) + R_{YY}(0) \geq \pm R_{XY}(\tau)$$

$$\therefore |R_{XY}(\tau)| \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$$

Hence proved

3) CCF Inequality:

$$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)\ R_{YY}(0)}$$

Proof: The CCF of $X(t)$ and $Y(t)$ is

$$= R_{XY}(\tau) = E[X(t)\ Y(t+\tau)]$$

We know $R_{xx}(0) = E[x^2(t)]$

$R_{yy}(0) = E[y^2(t)] = E[y^2(t+\tau)]$

let us consider $\left[\frac{x(t)}{\sqrt{R_{xx}(0)}} \pm \frac{y(t+\tau)}{\sqrt{R_{yy}(0)}}\right]^2 \geq 0$

$$\frac{x^2(t)}{R_{xx}(0)} + \frac{y^2(t+\tau)}{R_{yy}(0)} \pm 2\frac{x(t)\,y(t+\tau)}{\sqrt{R_{xx}(0)R_{yy}(0)}} \geq 0$$

$$E\left[\frac{x^2(t)}{R_{xx}(0)}\right] + E\left[\frac{y^2(t+\tau)}{R_{yy}(0)}\right] \pm 2\,E\left[\frac{x(t)\,y(t+\tau)}{\sqrt{R_{xx}(0)R_{yy}(0)}}\right] \geq 0$$

$$\frac{E[x^2(t)]}{R_{xx}(0)} + \frac{E[y^2(t+\tau)]}{R_{yy}(0)} \pm \frac{2\,E[x(t)\,y(t+\tau)]}{\sqrt{R_{xx}(0)R_{yy}(0)}} \geq 0$$

$$\frac{\cancel{R_{xx}(0)}}{\cancel{R_{xx}(0)}} + \frac{\cancel{R_{yy}(0)}}{\cancel{R_{yy}(0)}} \pm 2\frac{E[x(t)\,y(t+\tau)]}{\sqrt{R_{xx}(0)R_{yy}(0)}} \geq 0$$

$$\cancel{2} \geq \mp \frac{\cancel{2}\,E[x(t)\,y(t+\tau)]}{\sqrt{R_{xx}(0)R_{yy}(0)}}$$

(or)

$$\pm \frac{2R_{xy}(\tau)}{\sqrt{R_{xx}(0)R_{yy}(0)}} \geq -2$$

$$\mp \frac{R_{xy}(\tau)}{\sqrt{R_{xx}(0)R_{yy}(0)}} \geq -1$$

$$\mp \frac{R_{xy}(\tau)}{\sqrt{R_{xx}(0)R_{yy}(0)}} \leq 1$$

$$\therefore |R_{xy}(\tau)| \leq \sqrt{R_{xx}(0)R_{yy}(0)}$$

Hence proved.

4. Let the two random processes $x(t)$ & $y(t)$ having non-zero mean values and if they are ergodic random process with no periodic components then

$$\underset{|\tau|\to\infty}{Lt} R_{xy}(\tau) = \bar{X} \cdot Y$$

Proof:- The CCF b/w $X(t)$ & $Y(t) = R_{xy}(\tau) = E[X(t).Y(t+\tau)]$

If $X(t)$ & $Y(t)$ are ergodic random process

As $|\tau|\to\infty$ $X(t)$ & $Y(t)$ are no periodic Components then these are Considered as independent process

$$\underset{|\tau|\to\infty}{Lt} R_{XY}(\tau) = \underset{|\tau|\to\infty}{Lt} E[X(t)Y(t+\tau)]$$

$$E[X(t)Y(t+\tau)] = E[X(t)].E[Y(t+\tau)]$$

For ergodic process $E[X(t)] = \bar{X}$ & $E[Y(t+\tau)] = \bar{Y}$

$$\therefore \underset{|\tau|\to\infty}{Lt} R_{XY}(\tau) = E[X(t)]\,E[Y(t+\tau)]$$

$$\underset{|\tau|\to\infty}{Lt} R_{XY}(\tau) = \bar{X}\cdot\bar{Y}$$

5. Let two R.P's $X(t)$ & $Y(t)$ ≠ $R_{XY}(\tau)$ ~~$= E[X(t)Y(t+\tau)]$~~ having zero mean values with no periodic Components & they are ergodic process then $\underset{|\tau|\to\infty}{Lt} R_{XY}(\tau) = 0$

Proof:- The CCF b/w $X(t)$ & $Y(t)$ = $R_{XY}(\tau) = E[X(t)Y(t+\tau)]$

if $X(t)$ & $Y(t)$ are ergodic random process

As $|\tau|\to\infty$, $X(t)$ & $Y(t)$ are no periodic Components then these are Considered as independent process

$$\underset{|\tau|\to\infty}{Lt} R_{XY}(\tau) = \underset{|\tau|\to\infty}{Lt} E[X(t)Y(t+\tau)]$$

$$E[X(t)Y(t+\tau)] = E[X(t)]\,E[Y(t+\tau)]$$

For ergodic process, $E[X(t)] = \bar{X}$ & $E[Y(t+\tau)] = \bar{Y}$

$$\underset{|\tau|\to\infty}{Lt} R_{XY}(\tau) = E[X(t)]\,E[Y(t+\tau)]$$

$$= \bar{X}\cdot\bar{Y}$$

But $\overline{X} = 0$; $\overline{Y} = 0$

$\therefore \underset{|\tau| \to \infty}{Lt} R_{XY}(\tau) = 0$

6. Let two r.p's $X(t)$ & $Y(t)$ are independent and jointly WSS then $R_{XY}(\tau) = E[X(t)]\, E[Y(t+\tau)] = \overline{X} \cdot \overline{Y}$

Proof:- The CCF b/w $X(t)$ & $Y(t)$ is

$= R_{XY}(\tau) = E[X(t)\, Y(t+\tau)]$ if $X(t)$ & $Y(t)$ are jointly WSS

If $X(t)$ & $Y(t)$ are independent then

$$E[X(t)\, Y(t+\tau)] = E[X(t)]\, E[Y(t+\tau)]$$

$$R_{XY}(\tau) = E[X(t)]\, E[Y(t+\tau)]$$

If $X(t)$ & $Y(t)$ are joint WSS then

$$E[X(t)] = \overline{X} \quad \& \quad E[Y(t+\tau)] = \overline{Y}$$

$$\therefore R_{XY}(\tau) = E[X(t)]\, E[Y(t+\tau)]$$
$$= \overline{X} \cdot \overline{Y}$$

Auto Covariance function of random process:-

The auto Covariance function of R.P's is defined by

$$X(t) = C_{XY}(t, t+\tau) = E\{[X(t) - E[X(t)]]\,(X(t+\tau) - E[X(t+\tau)])\}$$

$$\therefore C_{XY}(x, y) = E[(X - E[X])(Y - E[Y])]$$

Proof:- The ACF of random process $X(t) = R_{XX}(t, t+\tau)$

$$= E[X(t)\, X(t+\tau)]$$

$$C_{XX}(t, t+\tau) = E\{X(t)\, X(t+\tau) - X(t)\, E[X(t+\tau)] - E[X(t)]\, X(t+\tau)$$
$$+ E[X(t)]\, E[X(t+\tau)]\}$$

$$C_{XX}(t, t+\tau) = E[X(t)\, X(t+\tau)] - E[X(t) \cdot E[X(t+\tau)]] - E[X(t)]\, E[X(t+\tau)]$$
$$+ E[X(t)]\, E[X(t+\tau)]$$

$E[X(t)]\,E[X(t+\tau)] = E[X(t+\tau)]\cdot E[X(t)]$

$C_{XX}(t, t+\tau) = R_{XX}(t, t+\tau) - E[X(t)]\,E[X(t+\tau)]$

properties:-

1. If $X(t)$ is WSS R.P's then $C_{XX}(\tau) = R_{XX}(\tau) - \overline{X}^2$

Proof:- The ACF of RP's $X(t) = C_{XX}(t, t+\tau) = R_{XX}(t, t+\tau) - E[X(t)]\,E[X(t+\tau)]$.

If $X(t)$ is WSS then

$R_{XX}(t, t+\tau) = E[X(t)\,X(t+\tau)] = R_{XX}(\tau)$

$E[X(t)] = \overline{X}$ & $E[X(t+\tau)] = \overline{X}$

$C_{XX}(t, t+\tau) = R_{XX}(\tau) - \overline{X}\cdot\overline{X}$

$= R_{XX}(\tau) - \overline{X}^2$

For WSS of $X(t)$; $C_{XX}(t, t+\tau) = C_{XX}(\tau)$.

$\therefore C_{XX}(\tau) = R_{XX}(\tau) - \overline{X}^2$

2. At $\tau = 0$; $C_{XX}(0) = \sigma_X^2 = Var(X)$

proof:- The ACF of RP's $X(t) = C_{XX}(t, t+\tau) =$

$R_{XX}(t, t+\tau) - E[X(t)]\,E[X(t+\tau)]$

If $X(t)$ is WSS, then

$R_{XX}(t, t+\tau) = E[X(t)\,X(t+\tau)] = R_{XX}(\tau)$

$E[X(t)] = \overline{X}$ & $E[X(t+\tau)] = \overline{X}$

$C_{XX}(t, t+\tau) = R_{XX}(\tau) - \overline{X}\cdot\overline{X}$

$= R_{XX}(\tau) - \overline{X}^2$

For WSS of $X(t)$; $C_{XX}(t, t+\tau) = C_{XX}(\tau)$

$C_{XX}(\tau) = R_{XX}(\tau) - \overline{X}^2$

put $\tau = 0$; $C_{XX}(0) = R_{XX}(0) - \overline{X}^2$

we know $R_{XX}(0) = \overline{X^2} = E[X^2(t)]$

$$C_{XX}(0) = \overline{X^2} - \overline{X}^2$$

$$Var(X) = \overline{X^2} - \overline{X}^2 = \sigma_X^2$$

3. The Auto Correlation Coefficient at $\tau = 0$ i.e $\rho_{XX}(0) = 1$

proof:- Auto Correlation Coefficient is defined by

$$\rho_{XY}(t, t+\tau) = \frac{C_{XX}(t, t+\tau)}{\sqrt{C_{XX}(t,t)\, C_{XX}(t+\tau, t+\tau)}}$$

At $\tau = 0$; $\rho_{XX}(t,t) = \dfrac{C_{XX}(t,t)}{\sqrt{C_{XX}(t,t)\, C_{XX}(t,t)}}$

$C_{XX}(t,t) = C_{XX}(0)$ & $\rho_{XX}(t,t) = \rho_{XX}(0)$

$$\therefore \rho_{XX}(0,0) = \frac{C_{XX}(0)}{\sqrt{C_{XX}(0)^2}} = 1$$

$$\therefore \rho_{XX}(0) = 1$$

Cross Covariance function of random process:-

The Cross Covariance function of R.P's $X(t)$ & $Y(t)$ is defined by

The CCF of $X(t)$ & $Y(t)$ $= C_{XY}(t, t+\tau) = E[X(t) - E[X(t)]]\, E[Y(t+\tau)]$

proof:- The CCF of $X(t)$ & $Y(t)$ $= R_{XY}(t, t+\tau) = E[X(t)\, Y(t+\tau)]$

$C_{XY}(t, t+\tau) = E[X(t)\, Y(t+\tau) - X(t)\, E[Y(t+\tau)]] - E[X(t)\, Y(t+\tau)]\, E[Y(t)\, E[X(t+\tau)]]$

$= E[X(t)\, Y(t+\tau)] - E[X(t)\, E[Y(t+\tau)]] - E[Y(t)\, E[X(t+\tau)]] +$

$E[Y(t)\, E[X(t+\tau)]]$

$$C_{XY}(t, t+\tau) = R_{XY}(t, t+\tau) - E[X(t)\, Y(t+\tau)]$$

Properties :-

1. If $x(t)$ & $y(t)$ are joint WSS R.P's then

$$C_{XY}(\tau) = R_{XY}(\tau) - \bar{X}\cdot\bar{Y}$$

Proof:- The Cross Covariance function of $X(t)$ & $Y(t)$

$$= R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)] = R_{XY}(\tau)$$

$$E[X(t)] = \bar{X} \quad \& \quad E[Y(t+\tau)] = \bar{Y}$$

$$C_{XY}(t, t+\tau) = R_{XY}(\tau) - \bar{X}\cdot\bar{Y}$$

$$\therefore C_{XY}(\tau) = R_{XY}(\tau) - \bar{X}\cdot\bar{Y}$$

Problems:-

i) The Auto Correlation function of random Variable $x(t)$ of $R_{XY}(\tau) = 36 + 25\, e^{-|\tau|}$. Find mean value, mean square value, average power & variance of R.P's $X(t)$?

Sol:-

We know $\underset{|\tau|\to\infty}{Lt} R_{XX}(\tau) = \bar{X}^2 = \left[E[X(t)]\right]^2$

$$\bar{X}^2 = \underset{|\tau|\to\infty}{Lt}\left[36 + 25\, e^{-|\tau|}\right]$$

$$= 36 + 25\, e^{-\infty}$$

$$\bar{X}^2 = 36$$

$$\therefore \bar{X} = \sqrt{36} = 6$$

$\therefore$ Mean value of $x(t) = \bar{X} = E[X(t)] = 6$

WKT, mean square value of $X(t) = E[X^2(t)] = \overline{X^2} = R_{XX}(0)$

$$\overline{X^2} = 36 + 25e^{-0}$$

$$= 36 + 25(1)$$

$$= 61$$

$$\therefore \overline{X^2} = 61$$

Average power of $X(t) = P_{avg} = \overline{X^2} = E[X^2(t)] = R_{XX}(0)$

$\therefore R_{XX}(0) = 61$ watts

Variance of $X(t) = \sigma_X^2 = \overline{X^2} - \overline{X}^2$

$= 61 - 36$

$= 25$

Standard deviation $= \sigma_X = \sqrt{25} = 5$

$\therefore \sigma_X = 5$.

2) Given $\overline{X} = 6$ and $R_{XX}(t, t+\tau) = 36 + 25e^{-\tau}$ for R.P's of $X(t)$. Indicate which of the following statements are true based on what is known with certainity $X(t)$.

(a) If first order stationary ?

(b) Has average power 61 Watts?

(c) Is WSS?

(d) Is Ergodic ?

(e) Has a periodic component.

(f) Has an ac power of 36 Watts.

Sol: a) Condition for first order stationary $X(t)$

i.e $E[X(t)] = \overline{X} =$ constant

$$\lim_{|\tau| \to \infty} R_{XX}(\tau)$$

$$= \lim_{|\tau| \to \infty} 36 + 25e^{-\tau}$$

$$= 36 + 25e^{-\infty}$$

$$= 36$$

$\therefore \overline{X}^2 = 36$

$\Rightarrow \overline{X} = 6 =$ constant.

Hence given $X(t)$ is first order stationary. R.P's.

(b) Mean square value of $X(t) = E[X^2(t)] = R_{XX}(0)$

$\tau \to 0$ $\quad \overline{X^2} = 36 + 25\, e^{-(0)}$

$= 36 + 25$

$\therefore \overline{X^2} = 61$ watts

Hence the given statement is true.

(c) Condition for WSS R.P's are (i) $E[X(t)] = \overline{X} =$ constant
(ii) $R_{XX}(t, t+\tau) = E[X(t)\, X(t+\tau)] = R_{XX}(\tau)$

Both conditions are satisfied so the given function is WSS Random process.

(d) $A[X(t)] = E[X(t)] =$ constant

$A[X(t)\, X(t+\tau)] = E[X(t)\, X(t+\tau)]$

Given ACF is not a function of time 't'. Therefore the above conditions are satisfied.

Hence given $X(t)$ is ergodic random process.

(e) $\underset{|\tau| \to \infty}{Lt} R_{XX}(\tau) = \overline{X}^2$

$\overline{X} = 6$, but given $\overline{X} = 6$

Hence $X(t)$ has no periodic components.

(f) AC power is nothing but variance of $X(t)$

$Var(X) = \sigma_X^2 = \overline{X^2} - \overline{X}^2 = 61 - 36 = 25$ watts

$(P_{avg})_{AC} = \sigma_X^2 = 25$ wats

Hence the given statement is false.

3) Let $X(t)$ be a stationary random process i.e differentiable denote its time derivative by $\dot{X}(t)$.

(a) S.T. $E[\dot{X}(t)] = 0$; (b) Find $R_{\dot{X}\dot{X}}(\tau)$ in terms of $R_{XX}(\tau)$

(a) let the radom process $X(t)$ & its time derivative $\dot{X}(t)$ is defined by

$$\dot{X}(t) = \underset{\Delta \to \infty}{Lt} \frac{X(t+\Delta) - X(t)}{\Delta}$$ ($\because \Delta$ is small incremental time)

Apply expectation we get

$$E[\dot{X}(t)] = E\left[\underset{\Delta \to \infty}{Lt} \frac{X(t+\Delta) - X(t)}{\Delta}\right]$$

$$= \underset{\Delta \to \infty}{Lt} \frac{E[X(t+\Delta) - X(t)]}{\Delta}$$

For WSS of $X(t)$, $E[X(t+\tau)] = E[X(t)] = \bar{X}$

$$E[X(t+\Delta)] = E[X(t)] = \bar{X}$$

$$E[\dot{X}(t)] = \underset{\Delta \to \infty}{Lt} \frac{\bar{X} - \bar{X}}{\Delta}$$

$$= \underset{\Delta \to \infty}{Lt} \frac{0}{\Delta}$$

$$= \underset{\Delta \to \infty}{Lt} (0)$$

$$\therefore E[\dot{X}(t)] = 0$$

(b) The ACF of random process $X(t)$

$$= R_{XX}(\tau) = E[X(t) \cdot X(t+\tau)]$$

The ACF of $\dot{X}(t)$

$$= R_{\dot{X}\dot{X}}(\tau) = E[\dot{X}(t)\,\dot{X}(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dot{x}(t)\, \dot{x}(t+\tau)\, f_{XX}(x_1, x_2 : t_1, t_2)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}(x(t)) \frac{\partial}{\partial t}(x(t+\tau))\, f_{XY}(x_1, x_2 : t_1, t_2)\, dx_1\, dx_2$$

$$= \frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\, x(t+\tau)\, f_{XX}(x_1, x_2 : t_1, t_2)\, dx_1\, dx_2$$

$$R_{\dot{X}\dot{X}}(\tau) = \frac{\partial^2}{\partial t^2}\left[R_{XX}(\tau)\right]$$

Home Work

1. The A.CF of R.V $X(t)$ is given by $R_{XX}(\tau) = 36 + \frac{16}{1+8\tau^2}$

Find mean value, mean square value, average power, Variance of the R.P's $X(t)$.

Sol: We know that $\underset{|\tau|\to\infty}{Lt} R_{XX}(\tau) = \bar{X}^2 = E[X(t)]^2$

$$\bar{X}^2 = \underset{|\tau|\to\infty}{Lt} \left(36 + \frac{16}{1+8\tau^2}\right)$$

$$= 36 + \frac{16}{1+\infty}$$

$$= 36 + 0$$

$$\bar{X}^2 = 36$$

$$\bar{X} = \sqrt{36} = 6$$

$\therefore$ Mean value of $X(t) = \bar{X} = 6$.

We know mean square value of $X(t) = E[X^2(t)]$

$$= \overline{X^2} = R_{XX}(0)$$

$\tau = 0$; $\overline{X^2} = 36 + \frac{16}{1+0}$

$$= 36 + 16$$

$$\therefore \overline{X^2} = 52$$

Average power of $X(t) = P_{avg} = \overline{X^2} = E[X^2(t)] = R_{XX}(0)$

$\therefore R_{XX}(0) = 52$ watts

Variance of $X(t) = Var(X) = \sigma_X^2 = \overline{X^2} - \bar{X}^2$

$$= 52 - 36$$

$$\sigma_X^2 = 12$$

2. The ACF of R.V $x(t)$ is given by $R_{xx}(\tau) = \frac{25\tau^2+36}{6.25\tau^2+4}$. Find mean value, mean square value, average power & variance of the R.P's $x(t)$.

Sol: We know that $\lim_{|\tau|\to\infty} R_{xx}(\tau) = \bar{X}^2 = E[x(t)]^2$

$$\text{Given } R_{xx}(\tau) = \frac{25\tau^2+36}{6.25\tau^2+4}$$

$$R_{xx}(\tau) = \frac{\tau^2\left(25+\frac{36}{\tau^2}\right)}{\tau^2\left(6.25+\frac{4}{\tau^2}\right)}$$

$$\bar{X}^2 = \lim_{|\tau|\to\infty} \frac{25+\frac{36}{\tau^2}}{6.25+\frac{4}{\tau^2}}$$

$$= \frac{25+0}{6.25+0}$$

$$\bar{X}^2 = \frac{25}{6.25} = 4$$

$$\therefore \bar{X} = \sqrt{4} = 2$$

$$\boxed{\bar{X} = 2}$$

We know meansquare value of $x(t) = E[x^2(t)] = \overline{X^2} = R_{xx}(0)$

$$\tau = 0; \quad \overline{X^2} = \frac{25\tau^2+36}{6.25\tau^2+4}$$

$$\overline{X^2} = \frac{25(0)+36}{6.25(0)+4}$$

$$\overline{X^2} = 9$$

Average power of $x(t) = P_{avg} = \bar{X}^2 = E[x^2(t)] = R_{xx}(0)$

$\therefore R_{xx}(0) = 4$ watts

Variance of $x(t) = Var(X) = \sigma_X^2 = \overline{X^2} - \bar{X}^2$

$$= 9 - 4$$

$$\sigma_X^2 = 5$$

3. The ACF of R.V. $x(t)$ is given by $R_{xx}(\tau) = 18 + \frac{2}{6+\tau^2}(1+4\cos(2\tau))$. Find mean value, mean square value, average power & variance.

Sol: We know that $\lim_{|\tau|\to\infty} R_{xx}(\tau) = \bar{X}^2 = E[x(t)]^2$

Given $R_{xx}(\tau) = 18 + \frac{2}{6+\tau^2}(1+4\cos(2\tau))$

$$\bar{X}^2 = \lim_{|\tau|\to\infty} 18 + \frac{2}{6+\tau^2}(1+4\cos(2\tau))$$

$$= 18 + \frac{2}{6+\infty}(1+4(\infty))$$

$$= 18 + \frac{2}{\infty}(1+\infty)$$

$$\therefore \bar{X}^2 = 18$$

$$\bar{X} = 3\sqrt{2} = 4.2426$$

Mean square value of $x(t) = E[x^2(t)] = \overline{X^2} = R_{xx}(0)$

$$\tau = 0 \; ; \; R_{xx}(0) = 18 + \frac{2}{6+0}(1+4)$$

$$\overline{X^2} = 18 + \frac{2}{6}(5)$$

$$= \frac{59}{3}$$

$$\therefore \overline{X^2} = 19.667$$

Average power of $x(t) = \overline{X^2} = E[X^2(t)] = R_{xx}(0)$

$\therefore R_{xx}(0) = 19.667$ watts

Variance of $x(t) = Var(X) = \sigma_X^2 = \overline{X^2} - \bar{X}^2$

$$= 19.667 - 18$$

$$= 1.667$$

4. Define a random process $X(t) = A\cos(\pi t)$ where A is gaussian random variable with zero mean & variance σ_a^2 is $X(t)$ is stationary in any sense.

Sol, Given that $X(t) = A\cos(\pi t)$

where A is gaussian random variable with zero mean & variance σ_a^2.

$\therefore$ The PDF of gaussian random variable $= f_A(A)$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_a} e^{-\frac{A^2}{2\sqrt{a^2}}}$$

Conditions for stationary random process (WSS)

(i) $E[X(t)] = \bar{X} = \text{constant}$

(ii) $R_{XX}(t, t+\tau) = E[X(t)\cdot X(t+\tau)] = R_{XX}(\tau)$

Mean value of $X(t) = E[X(t)] = \bar{X} = \int_{-\infty}^{\infty} X(t)\, f_A(A)\, dA$.

$$= \int_{-\infty}^{\infty} A\cos(\pi t) \cdot \frac{1}{\sqrt{2\pi}\sqrt{a}} e^{-\frac{A^2}{2\sqrt{a^2}}}\, dA$$

$$= \frac{\cos(\pi t)}{\sqrt{2\pi}\sqrt{a}} \int_{-\infty}^{\infty} \underbrace{\left(A \cdot e^{-\frac{A^2}{2\sqrt{a^2}}}\right)}_{\text{odd function}} dA$$

For odd function integral is 0, i.e. $\frac{\cos \pi t}{\sqrt{2\pi}\sqrt{a}}(0)$

$E[X(t)] = 0 = \text{constant}$

$$R_{XX}(t, t+\tau) = E[X(t)\, X(t+\tau)] = \int_{-\infty}^{\infty} X(t)\, X(t+\tau)\, f_A(A)\, dA$$

$$= \int_{-\infty}^{\infty} A\cos(\pi t)\, A\cos(\pi(t+\tau)) \cdot \frac{1}{\sqrt{2\pi}\,\sigma_a} e^{-\frac{A^2}{2\sigma_a^2}}\, dA$$

$$= \frac{\cos(\pi t)\cos(\pi(t+\tau))}{\sqrt{2\pi}\,\sigma_a} \int_{-\infty}^{\infty} \underbrace{A^2 e^{-\frac{A^2}{2\sigma_a^2}}}_{\text{even function}} dA$$

$$= \frac{\cos(\pi t)\cos\pi(t+\tau)}{\sqrt{2\pi}\,\sigma a} \cdot 2\int A^2 e^{\frac{-A^2}{2\sigma a^2}} dA$$

function of 't' $\qquad$ constant value

$\therefore \; R_{xx}(t, t+\tau) \neq R_{xx}(\tau)$

i.e, ACF is a function of time t. Hence the given random process x(t) is not a stationary random process.

5). A random process $X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$ where ω_0 is constant and A & B are random variables. If A and B are uncorrelated zero mean having same variances σ^2 but different density functions then show that X(t) is a WSS.

Sol: Given random process $X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$ where ω_0 is constant and A & B are random variables having zero mean & same variance σ^2.

A and B are uncorrelated R.V's. i.e. $E[AB]=0$

Mean value of $A = E[A] = 0$; $Var(A) = \sigma^2$

Mean value of $B = E[B] = 0$; $Var(B) = \sigma^2$

$$Var(A) = E[A^2] - (E(A))^2$$

$$\sigma^2 = E[A^2] - 0$$

$$\sigma^2 = E[A^2]$$

$$\boxed{\therefore \; E[A^2] = \sigma^2}$$

$$Var(B) = E[B^2] - (E[B])^2$$

$$\sigma^2 = E[B^2] - 0$$

$$\sigma^2 = E[B^2]$$

$$\boxed{\therefore \; E[B^2] = \sigma^2}$$

Condition for WSS are

i) $E[X(t)] = \bar{X} = \text{constant}$

(ii) $R_{xx}(t, t+\tau) = E[X(t) \cdot X(t+\tau)] = R_{xx}(\tau)$

$$E[X(t)] = E[\underset{\text{R.V's}}{A} \cos(\underset{\text{constant}}{\omega_0 t}) + B \sin(\omega_0 t)]$$

$$= E[A] \cos\omega_0 t + E[B] \sin\omega_0 t$$

$$= 0 + 0$$

$$\therefore E[X(t)] = 0 = \text{constant}$$

$$R_{xx}(t, t+\tau) = E[X(t) \cdot X(t+\tau)] = R_{xx}(\tau)$$

$$E[X(t) X(t+\tau)] = E[(A\cos\omega_0 t + B\sin\omega_0 t)(A\cos\omega_0(t+\tau) + (B\sin(\omega_0(t+\tau)))]$$

$$= E[A^2 \cos\omega_0 t \cos(\omega_0 t + \omega_0\tau) + AB \cos\omega_0 t \sin(\omega_0 t + \omega_0\tau)$$

$$+ AB \sin\omega_0 t \cos(\omega_0 t + \omega_0\tau) + B^2 \sin(\omega_0 t) \sin(\omega_0 t + \omega_0\tau)]$$

$$= \sigma^2 [\cos\omega_0 t \cos(\omega_0 t + \omega_0\tau)] + \sin\omega_0 t \sin(\omega_0 t + \omega_0\tau)$$

$$= \sigma^2 \cos(\omega_0 t - \omega_0 t - \omega_0\tau) \quad \because \cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$= \sigma^2 \cos(\omega_0\tau)$$

$R_{xx}(t, t+\tau) = R_{xx}(\tau)$

$\therefore$ The above two conditions are satisfied with r.v.

Hence the given R.P's is a WSS R.P's.

6. let two random variables $X(t) = A\cos\omega_0 t + B\sin\omega_0 t$, $Y(t) = B\cos\omega_0 t - A\sin\omega_0 t$ where ω_0 is constant and A and B are random variables. If A & B are uncorrelated zero mean having same variance σ^2 but different density function then show that $X(t)$ & $Y(t)$ are jointly WSS.

Sol: Given random variables are $X(t) = A\cos\omega_0 t + B\sin\omega_0 t$

$$Y(t) = B\cos\omega_0 t - A\sin\omega_0 t$$

Conditions for WSS are

i) $E[X] = \bar{X} = \text{constant}$

ii) $E[X(t)] = E[A\cos\omega_0 t + B\sin\omega_0 t]$

$$= E[A]\cos\omega_0 t + E[B]\sin\omega_0 t$$

$$= 0 + 0$$

$$= 0$$

$E[X(t)] = \text{constant}$

6. Check the following for WSS

(i) $R_{xx}(t, t+\tau) = \cos(t\, e^{-|t+\tau|})$

Sol: Conditions of WSS random process.

(a) $E[X] = \bar{X} =$ constant $\bar{X} = 1$

(b) $R_{xx}(t, t+\tau) = E[x(t) \cdot x(t+\tau)] = R_{xx}(\tau)$

Given ACF $R_{xx}(t, t+\tau)$ is a function of absolute time t. i.e., $R_{xx}(t, t+\tau) \neq R_{xx}(\tau)$. Hence, the given $x(t)$ is not an WSS random process.

(ii) $R_{xx}(t, t+\tau) = \sin\left(\frac{2\tau}{1+\tau^2}\right)$ $\bar{X} = 0$

Sol: Given ACF $R_{xx}(t, t+\tau)$ is not a function of t, i.e. $R_{xx}(t, t+\tau) = R_{xx}(\tau)$. Hence, the given $x(t)$ is a WSS random process.

(iii) $R_{xx}(t, t+\tau) = -10^{-|\tau|} \cancel{+10^{-|\tau|}}$ $\bar{X} = 0$

Sol: Given ACF $R_{xx}(t, t+\tau)$ is not a function of t, i.e. $R_{xx}(t, t+\tau) = R_{xx}(\tau)$. Hence, the given $x(t)$ is a WSS random process.

(iv) $R_{xx}(t, t+\tau) = 5\, e^{-|\tau|}$ $\bar{X} = 0$

Sol: Given ACF $R_{xx}(t, t+\tau)$ is not a function of 't', i.e., $R_{xx}(t, t+\tau) = R_{xx}(\tau)$. Hence, the given $x(t)$ is a WSS random process.

7. Given the ACF for a stationary ergodic process with no ~~ergodic~~ periodic components is

$R_{xx}(y) = 25 + \frac{4}{1+6y^2}$

Find the mean value and variance of $x(t)$

Sol: Given $R_{xx}(y) = 25 + \frac{4}{1+6y^2}$

let $y = \tau$

$$R_{xx}(\tau) = 25 + \frac{4}{1+6\tau^2}$$

Mean value of $x(t) = \underset{|\tau|\to\infty}{Lt} R_{xx}(\tau) = \overline{X}^2 = \underset{|\tau|\to\infty}{Lt}\left(25 + \frac{4}{1+6\tau^2}\right)$

$$= 25 + \frac{4}{1+6(\infty)^2}$$

$$\overline{X}^2 = 25$$

$$\boxed{\therefore \overline{X} = 5}$$

Mean square value of $x(t) = \overline{X^2(t)} = \overline{X^2} = E[x^2(t)] = R_{xx}(\tau)$

$$= 25 + \frac{4}{1+6(0)}$$

$$= 25 + 4$$

$$\overline{X^2(t)} = 29$$

Variance of $x(t) = \sigma_X^2 = \overline{X^2} - \overline{X}^2$

$$= 29 - (5)^2$$

$$= 29 - 25$$

$$\boxed{\therefore \sigma_X^2 = 4}$$

8. A stationary random process $x(t)$ with mean 2 has the ACF, $R_{xx}(\tau) = 4 + e^{-|\tau|/10}$. Find the mean and variance of $y = \int_0^1 x(t)\,dt$.

Sol: Given mean value of $x(t) = \overline{X} = 2 = E[x(t)]$

$$ACF,\ R_{xx}(\tau) = 4 + e^{-|\tau|/10}$$

New random variable $y = \int_0^1 x(t)\,dt$

Mean value of $Y = E[Y(t)] = E\left[\int_0^1 x(t)\, dt\right]$

$$= E\left[\int_0^1 E\,x(t)\, dt\right]$$

$$= \left(\int_0^1 2\, dt\right)$$

$$= 2\,[t]_0^1$$

$$= 2\,[1-0]$$

$$\boxed{\therefore E[Y(t)] = 2}$$

Mean square value of $Y = ?$

let $Y = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, dt = \frac{2}{T}\int_{-T}^{T} x(t)\, dt = \frac{1}{T}\int_0^T x(t)\, dt$ then

$$\overline{Y^2} = E[Y^2] = \frac{1}{T}\int_{-T}^{T}\left[1-\frac{|\tau|}{T}\right] R_{XX}(\tau)\, d\tau$$

Given $\overset{T=1}{Y} = \int_0^1 x(t)\, dt \longrightarrow \overline{Y^2} = E[Y^2(t)] = \int_{-1}^{1}\left[1-\frac{|\tau|}{T}\right] R_{XX}(\tau)\, d\tau$

$$\overline{Y^2} = \int_{-1}^{1}\left(1-\frac{|\tau|}{1}\right)\left(4+e^{-|\tau|/10}\right) d\tau$$

$$\because |\tau| = \begin{cases} \tau\ ; & \tau \geq 0 \\ -\tau & \tau < 0 \end{cases}$$

$$= \int_{-1}^{0}(1-(-\tau))\left(4+e^{-(-\tau)/10}\right) d\tau + \int_0^1 (1-\tau)\left(4+e^{-\tau/10}\right) d\tau$$

$$= \int_{-1}^{0}(1+\tau)\left(4+e^{\tau/10}\right) d\tau + \int_0^1 (1-\tau)\left(4+e^{-\tau/10}\right) d\tau$$

* In an experiment of rolling a die and flipping a coin the random variable (X) is chosen such that

(i) A coin Head (H) outcome that corresponds to positive value of X that are equal to the number that shown on die

(ii) A coin tail (T) outcome corresponds to negative value of X that are equal in magnitude to twice the number that shown on die. Map the elements of random variable (X) into points on the real line and explain.

Sol: If the coin shows head, the numbers on die are taken as positive.

If the coin shows tail, the numbers on die are taken as negative.

In the combined experiment, if H shows head and T shows tail then, the set of events in the samples is

$$S = \{(T,-12)\ (T,-10),(T,-8)\ (T,-6)\ (T,-4)\ (T,-2)\ (H,1)\ (H,2)\ (H,3)\ (H,4)(H,5)(H,6)\}$$

Then the mapping of the sets to X as

$$X = \{-12,-10,-8,-6,-4,-2,1,2,3,4,5,6\}$$

These values are then the elements on the real line axis as shown in figure.

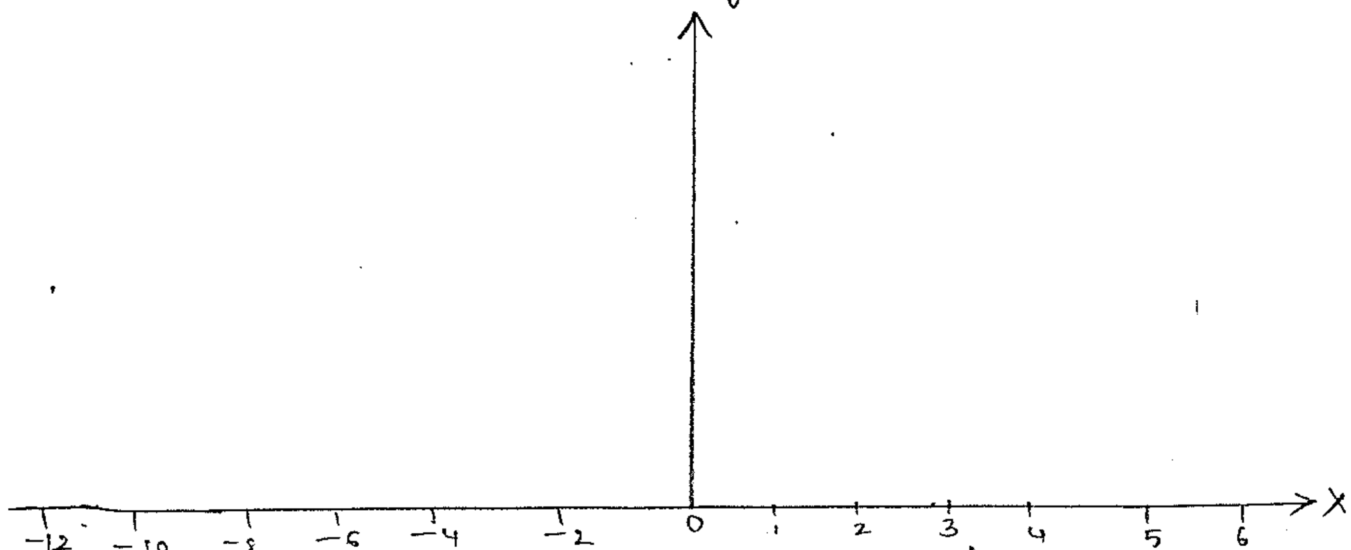


Fig: Mapping of R-V on real axis

Here a random variable X is a function that maps each point in S into a point on the real line. It may be many-to-one mapping.

* In an experiment, where the pointer on wheel of chance is spun, the possible outcomes are the numbers from 0 to 12 marked on the wheel. The sample space consists of the number in the set $\{0 < s \leq 12\}$. If the r.v. X is defined as $X = X(s) = s^2$, map the elements on the r.v on the real line and explain.

Sol: Here the experiment is the pointer on the wheel of chance is spun. The possible outcomes of the experiment are the numbers from 0 to 12 i.e., the sample space $S = \{0 < s \leq 12\} = \{1 \leq s \leq 12\}$

A random variable is given as $X = X(s) = s^2 = \{1 \leq X \leq 144\}$
$= \{0 < X \leq 144\}$

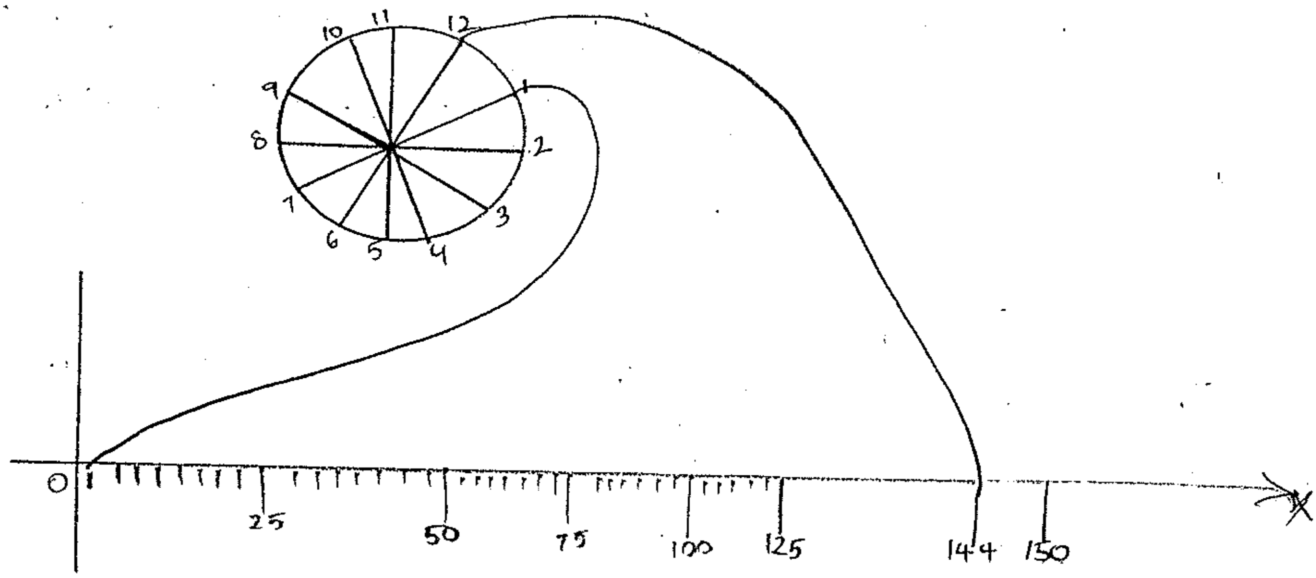


Fig: Mapping of S to a real line.

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Unit - 5

Random Process Spectral Characteristics

Power spectral density (PSD) (or) Power spectrum Density (PDS):

Definition ①: The power density spectrum of R.P's $X(t)$ is given by i.e.,

$$X(t) = S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{E[|X_T(\omega)|]^2}{2T}$$

Here $X_T(\omega)$ is fourier ~~spectrum~~ transform of $X(t)$ in the interval $[-T, T]$.

Definition ②: The power spectrum density of R.P's $X(t)$ is defined as the fourier transform of autocorrelation function of R.P's $X(t)$

$$\text{PSD of } X(t) = S_{XX}(\omega) = F[R_{XX}(\tau)] = \int_{-\infty}^{\infty} [R_{XX}(\tau)] e^{-j\omega\tau} d\tau \quad \rightarrow ①$$

The auto correlation function of R.P's $X(t)$ is defined as the inverse fourier transform of power spectral density of R.P's $X(t)$ i.e.,

$$\text{The ACF of } X(t) = R_{XX}(\tau) = F^{-1}[S_{XX}(\omega)]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} [S_{XX}(\omega)] e^{j\omega t} d\omega \longrightarrow ②$$

above eq ① and ② are called as "Wiener- Khinchine relations".

$\therefore$ ACF & power spectrum density are fourier transform pairs.

Properties of Power Spectrum Density:

* let $X(t)$ is a WSS random process, then PSD of $X(t)$ satisfies the following properties.

1. PSD is always non-negative i.e $S_{XX}(\omega) \geq 0$

Proof: The PSD of $X(t) = S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{E[|X_T(\omega)|^2]}{2T}$